On Decoding Random-Access SC-LDPC Codes

Eshed Ram Yuval Cassuto
Andrew and Erna Viterbi Department of Electrical Engineering
Technion – Israel Institute of Technology, Haifa 32000, Israel
E-mails: {s6eshedr@campus, ycassuto@ee}.technion.ac.il

***Abstract*—We study a new decoding strategy of multi-block SC-LDPC codes motivated by data-storage applications. To decode a sub-block out of the full code block, our proposed decoder accesses a small number of sub-blocks around the desired sub-block. We call this decoding strategy "semi-global decoding", and parametrize it by its access cost: the number of accessed sub-blocks. We provide a theoretical characterization of decoding performance, and evaluate this performance for random-access SC-LDPC ensembles.**

1. Introduction

Spatially-coupled low-density parity-check (SC-LDPC) codes [1] were extensively studied recently, and were shown to have many desired properties. For example, threshold saturation [2], linear-growth of minimum distance [3] and linear-growth of minimal trapping sets of typical codes from the ensemble [4], imply good BER performance in the *waterfall* and *error floor* regions, using the Belief-Propagation (BP) decoder. Moreover, the special structure of SC-LDPC codes, where bits participating in a particular parity-check equation are spatially close to each other, renders a locality property that can be exploited to implement low-latency high-throughput belief-propagation based decoders; such decoders are pipelined decoders [1], [5] and window decoders [7], [8], [6].

When used in data-storage applications, where decoding failures imply data-losses, an error-correcting code must protect against extremely high noise levels (although most noise instances are much milder), requiring very large block lengths and complex decoding. A possible solution to this problem are random-access codes [9], [10], [11], where (small) sub-blocks of a long block can be decoded (local decoding) independently from each other for fast read access. In case of catastrophic error events, the full code block is decoded (global decoding) for increased data reliability. In this paper we study SC-LDPC codes decoded in a mode in between local and global decoding, which we call *semi-global (SG) decoding*. A SG decoder decodes a requested *target* sub-block using its neighboring sub-blocks, which we call *helper* sub-blocks. Our work follows up on [11] where we introduced protograph-based random-access SC-LDPC codes called SC-LDPCL codes. Semi-global decoding was suggested in [11], but only with "black-box" analysis based on the single sub-block threshold.

In this paper we develop an analysis for true semi-global decoding, which accounts for propagation of partial information between neighboring sub-blocks. We define the SG decoding graphs, and introduce a density-evolution framework to analyze them. The new analysis reveals that decoding a target sub-block using helper sub-blocks from both sides is superior to the one-sided variant proposed in [11]. We compare between global and semi-global decoding over the BEC, exemplifying that the threshold loss is negligible, while the complexity reduction is substantial. We further evaluate the performance of semi-global decoding over a channel with variability in the sub-block erasure parameter.

2. Preliminaries

A. Protograph Based SC-LDPC Codes

An LDPC protograph is a (small) bipartite graph $\mathcal{G} = (\mathcal{V} \cup \mathcal{C}, \mathcal{E})$, where $\mathcal{V}, \mathcal{C}, \mathcal{E}$ are the sets of variable nodes (VNs), check nodes (CNs), and edges, respectively. A protograph is said to be (l, r)-regular if for every VN $v \in \mathcal{V}$ has degree l, and every CN $c \in \mathcal{C}$ has degree r. A Tanner graph is generated from a protograph $\mathcal{G}$ by a lifting ("copy-and-permute") operation specified by a lifting parameter L (for more details see [3]). If we let $L \to \infty$, then we can analyze the performance of the BP decoder on the resulting ensemble of Tanner graphs via density evolution (DE) on the original protograph $\mathcal{G}$, and calculate its asymptotic decoding threshold. For the binary erasure channel (BEC), we can compute the threshold $\epsilon^*(\mathcal{G})$. A protograph $\mathcal{G} = (\mathcal{V} \cup \mathcal{C}, \mathcal{E})$ can be represented through a bi-adjacency matrix $H_{\mathcal{G}}$, where the VNs in $\mathcal{V}$ are indexed by the columns of $H_{\mathcal{G}}$, and the CNs in $\mathcal{C}$ by the rows.

An (l, r)-regular spatially-coupled (SC) LDPC protograph is constructed by coupling together (l, r)-regular protographs, in the following way. Let $B = 1^{l \times r}$ be an all-ones matrix representing an (l, r)-regular LDPC protograph, and let $\{B_t\}_{t=1}^{T}$ be T matrices such that $B = \sum_{t=1}^{T} B_t$ (T is the *coupling width*). Coupling M copies of B amounts to diagonally placing copies of $\left(B_1; B_2; \ldots; B_T\right)$ as in Figure 1 (b) (see [3] and references therein).

In the rest of the paper, we focus on (l, r)-regular SC-LDPC protographs with a coupling width $T = 2$. The results can be readily extended to irregular base matrices and a larger coupling width, albeit with a more cumbersome notation.

Example 1. *A $(3, 6)$-regular SC-LDPC protograph is obtained with $T = 2$, $B_1 = (1\,1\,0\,0\,0\,0\,; 1\,1\,1\,1\,0\,0\,; 1\,1\,1\,1\,1\,1)$ and $B_2 = 1^{3 \times 6} - B_1$.*

B. Random-Access SC-LDPC Codes: review from [11]

Our new semi-global decoder relies on a construction from [11] of SC-LDPC codes with sub-block access, which we review in this sub-section. Consider a coupled protograph $\mathcal{G} = (\mathcal{V} \cup \mathcal{C}, \mathcal{E})$, divided into $M > 1$ sub-blocks (SBs) $\{\mathcal{V}_m\}_{m=1}^{M}$. In what follows, let $H = H_{\mathcal{G}}$ be a bi-adjacency matrix representing the coupled protograph $\mathcal{G}$, and let $m \in \{1, 2, \ldots, M\}$ be a SB index. When decoding SB m locally, all of the bits outside the SB are treated as erasures; hence only

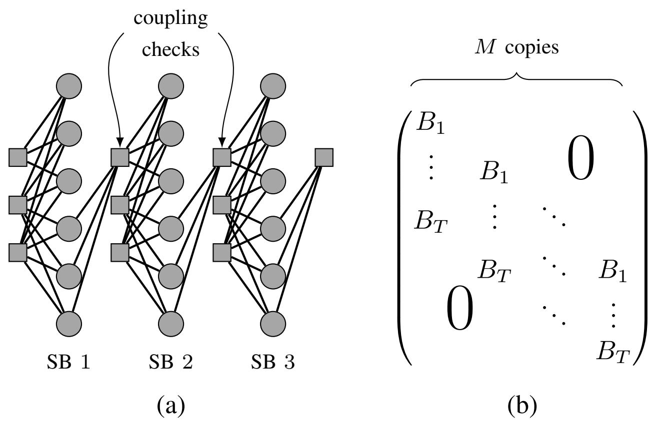

Fig. 1. (a) The $(3, 6, 1)$ SC-LDPCL protograph with $M = 3$ SBs; (b) the infinite matrix representing the protograph coupling operation.

CNs connected inside SB m are relevant to local decoding. The next definition formalizes this observation.

Definition 1.
1) *If VN $j \in \mathcal{V}$ belongs to SB m, we write $j \in \mathcal{V}_m$.*
2) *CN $i \in \mathcal{C}$ is said to be a local check (LC) in SB m if and only if $\{j : H_{i,j} = 1\} \subseteq \mathcal{V}_m$, and we write $i \in \mathcal{C}_m$.*
3) *CN $i \in \mathcal{C}$ is said to be a coupling check (CC) if it is not a local check.*
4) *The local protograph of SB m is the sub-graph $\mathcal{G}_m = (\mathcal{V}_m \cup \mathcal{C}_m, \mathcal{E}_m)$, where $\mathcal{E}_m$ is the set of edges in $\mathcal{E}$ that connect between $\mathcal{V}_m$ and $\mathcal{C}_m$.*
5) *The global and local BP decoding thresholds are given by $\epsilon^*_G \triangleq \epsilon^*(\mathcal{G})$ and $\epsilon^*_m \triangleq \epsilon^*(\mathcal{G}_m)$, respectively.*

The following construction gives SC-LDPC codes with non-zero local decoding thresholds, which we call *random-access SC-LDPC* codes, or for short, *SC-LDPCL* (suffix L stands for locality) codes.

Construction 1 ((l, r, t) SC-LDPCL)**.** *Let $t \in \{1, \dots, l-1\}$, and let A_1 be a $t \times r$ matrix given by*

$$A_1 = \begin{pmatrix} \underline{1} & & & & & \underline{0} \\ & & & 0 & & \\ \underline{1} & \underline{1} & & & & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} & & & \underline{0} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \underline{1} & \underline{1} & \underline{1} & \cdots & \underline{1} & \underline{0} \end{pmatrix},$$

where $\underline{1}$ and $\underline{0}$ are length-$\lfloor \frac{r}{t+1} \rfloor$ all-ones and length-$\left(r - t\lfloor \frac{r}{t+1} \rfloor\right)$ all-zeros vectors, respectively. Let A_2 be an all-ones $(l-t) \times r$ matrix. We build the (l, r, t) protograph as in Figure 1 (b) with $T = 2$, and M copies of $\left(B_1; B_2\right)$ on the diagonal, where $B_1 = \left(A_1; A_2\right)$ and $B_2 = 1^{l \times r} - B_1$.

The parameter t specifies the number of check nodes connecting two adjacent sub-blocks (i.e., there are $l - t$ local check nodes). In fact, t controls the local-global trade-off of the protograph (see [11] for more details).

Example 2. *Figure 1 (a) illustrates the $(3, 6, 1)$ random-access SC-LDPCL protograph with $M = 3$ SBs.*

For the rest of the paper, we denote the set of integers $\{a, a+1, \dots, b\}$ by $[a : b]$.

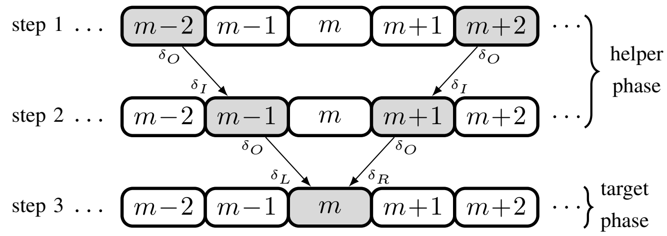

Fig. 2. Example of SG decoding with target SB $m \in [1 : M]$, and $d = 4$; the steps are shown from top to bottom. The gray SBs are those that are decoded in a given step, and the arrows represent information passed between sub-blocks.

3. Semi-Global Decoding

In this section we define semi-global (SG) decoding, and analyze it through DE. As shown later, SG decoding has a substantial complexity advantage over global decoding with a very small cost in threshold. Consider a SC-LDPCL protograph with M SBs, and assume that the user wants to extract SB $m \in [1 : M]$. We call SB m the *target*. In SG decoding, the decoder uses d *helper* SBs to decode the target in two phases: the *helper phase*, and the *target phase*. In the former, helper SBs are decoded locally using information from other previously-decoded helper SBs, and in the latter, the target SB is decoded using information from its neighboring helper SBs.

Example 3. *Figure 2 exemplifies SG decoding with $d = 4$ helper SBs. The helper phase consists of decoding helpers m–2 and m+2 locally, and decoding helpers $m-1$ and $m+1$ using the information from helpers $m-2$ and $m+2$, respectively. In the target phase, SB m is decoded using information from both SB $m-1$ and $m+1$.*

Semi-global decoding resembles window decoding of SC-LDPC codes [8], but differs in: 1) there is no overlap between two window positions, which decreases latency, and 2) decoding starts close to the target SB (i.e. not necessarily at the first or last SBs), thus allowing low-latency access to sub-blocks anywhere in the block. The built-in structure of SC-LDPCL protographs enables these two distinctions.

The complexity reduction of SG decoding, compared to global decoding, comes from both specifying $d < M$, and the fact that decoder messages between sub-blocks are exchanged in one direction only. To see this, consider the $(3, 6, 1)$ SC-LDPCL protograph (Figure 1(a)), and assume SG-decoding of target SB 2 with helpers SBs 1 and 3. In the helper phase, we decode SB 1 and 3 locally, so the coupling checks are erased, and the decoder ignores all edges connected to them. In the target phase, the coupling checks are no longer erased, but they send information towards the target SB only. As a result, the 6 edges connecting the coupling checks to SBs 1 and 3 do not participate in SG decoding.

A. SG Density Evolution

We now perform DE analysis for helper and target SBs during SG erasure decoding. We denote the incoming (resp. outgoing) edges carrying messages to (resp. form) a helper SB by $\underline{\delta}_I$ (resp. $\underline{\delta}_O$). The incoming messages to the target SB from the left-side and right-side helper SBs are denoted by $\underline{\delta}_L$ and $\underline{\delta}_R$, respectively. Note that $\underline{\delta}_O$ of some helper is either $\underline{\delta}_I$ of the next helper, or one of the incoming messages to the target, $\underline{\delta}_R$ or $\underline{\delta}_R$ (see Figure 2). With a little abuse of notations, we use $\underline{\delta}_I\, \underline{\delta}_O, \underline{\delta}_L, \underline{\delta}_R$ to mark the erasure probabilities carried on these edges.

Definition 2. *Let $\mathcal{G}$ be an (l, r, t) SC-LDPCL protograph, and let $w \triangleq \left\lfloor \frac{r}{t+1} \right\rfloor$. The* semi-global graph $\mathcal{G}_{SG} = (\mathcal{V} \cup \mathcal{C}, \mathcal{E})$ *is a bipartite graph equipped with a VN labeling function $L_{\mathcal{V}} \colon \mathcal{V} \to [1 : t+1]$, an edge labeling function $L_{\mathcal{E}} \colon \mathcal{E} \to [1 : (t+1)^2]$, and $2t$ incoming edges $\{\delta_{R,1}, \dots, \delta_{R,t}\}$, and $\{\delta_{L,1}, \dots, \delta_{L,t}\}$ such that:*

1) *$\mathcal{V} = \{v_1, v_2, \dots, v_r\}$ is a set of r VNs.*
2) *$\mathcal{C}$ is a set of $l+t$ CNs: $l-t$ of them are local checks (LCs), t of them are right coupling checks (RCCs), and another t are left coupling checks (LCCs).*
3) *We mark the $2t$ coupling checks as $c_{R,1}, \dots, c_{R,t}$, and $c_{L,1}, \dots, c_{L,t}$. For every $i \in [1 : t]$, $c_{R,i}$ (resp. $c_{L,i}$) is connected to an incoming edge $\delta_{R,i}$ (resp. $\delta_{L,i}$).*
4) *There is one edge between every LC and every VN. For every $i \in [1 : t]$, $c_{R,i}$ is connected to $r - iw$ VNs, one edge to each VN v_j, where $j \in [1 + iw : r]$, and for every $i \in [1 : t]$, $c_{L,i}$ is connected to iw VNs, one edge to each VN v_j, where $j \in [1 : iw]$.*
5) *For every $k \in [1 : t+1]$ and every VN $v \in \mathcal{V}$, we label $L_{\mathcal{V}}(v) = k$ if v is connected to $k-1$ RCCs.*
6) *For every edge $e = (v, c) \in \mathcal{E}$, if $L_{\mathcal{V}}(v) = k$, we label*
$$L_{\mathcal{E}}(e) = \begin{cases} k & c \text{ is an LC} \\ s_{k,t} + i & c = c_{R,i} \\ v_{k,t} + i & c = c_{L,i} \end{cases},$$
where $s_{k,t} \triangleq t+1+\frac{(k-1)(k-2)}{2}$, and $v_{k,t} \triangleq 2t+1+\frac{t(t-1)}{2}+\frac{(k-1)(2t-k)}{2}$

Remark. *Definition 2 refers to the graph when decoding the target SB. The helper graph is similar with the only difference that t incoming edges (say, $\{\delta_{L,1}, \dots, \delta_{L,t}\}$) turn into outgoing edges $\{\delta_{O,1}, \dots, \delta_{O,t}\}$, and the checks connected to them do not participate during the SB's decoding.*

Example 4. *Figure 3 illustrates the semi-global graph (helper and target) corresponding to the $(l = 3, r = 6, t = 1)$ SC-LDPCL protograph from Example 2. Dotted edges in the helper graph do not participate during decoding, and are calculated at the end of the step to output a message on δ_O. There are $t+1 = 2$ VN labels (drawn inside nodes), and $(t+1)^2 = 4$ edge labels (drawn on edges) in the target graph: red, blue, violet and black for edge types 1, 2, 3, and 4, respectively.*

Since edges with same labels are connected to VNs and CNs of the same degree, then in terms of density evolution, in any iteration, the erasure fraction of edges $e_1, e_2 \in \mathcal{E}$ coincide if $L_{\mathcal{E}}(e_1) = L_{\mathcal{E}}(e_2)$. This structure is the key observation for reducing the complexity of the DE equations. Due to space limitations we limit the presentation of DE equations to the special case of $t = 1$, but similar equations can be obtained with complexity growing quadratically with t.

B. The $t = 1$ Case

In view of Definition 2, if $t = 1$, then there are 4 different edge types in the target SB graph. However, as shown below,

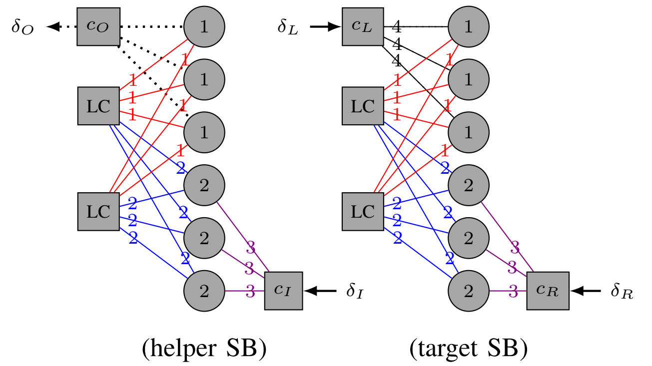

Fig. 3. The $(3, 6, 1)$ semi-global graphs $\mathcal{G}_{SG}$ as described in Definition 2.

it is sufficient to track only 2 edge types. This simplification renders a two-dimensional graphical representation of the DE equations. We define $x_\ell^{(k)}$ as the erasure fraction carried on any edge with label k in iteration ℓ. Then,

$$x_\ell^{(1)} = \epsilon \left[1 - \left(1 - x_{\ell-1}^{(1)}\right)^{w-1} \left(1 - x_{\ell-1}^{(2)}\right)^{r-w}\right]^{l-2} \cdot \left[1 - \left(1 - x_{\ell-1}^{(4)}\right)^{w-1} (1 - \delta_L)\right], \quad \text{(1a)}$$

$$x_\ell^{(2)} = \epsilon \left[1 - \left(1 - x_{\ell-1}^{(1)}\right)^{w} \left(1 - x_{\ell-1}^{(2)}\right)^{r-w-1}\right]^{l-2} \cdot \left[1 - \left(1 - x_{\ell-1}^{(3)}\right)^{r-w-1} (1 - \delta_R)\right], \quad \text{(1b)}$$

$$x_\ell^{(3)} = \epsilon \left[1 - \left(1 - x_{\ell-1}^{(1)}\right)^{w} \left(1 - x_{\ell-1}^{(2)}\right)^{r-w-1}\right]^{l-1}, \quad \text{(1c)}$$

$$x_\ell^{(4)} = \epsilon \left[1 - \left(1 - x_{\ell-1}^{(1)}\right)^{w-1} \left(1 - x_{\ell-1}^{(2)}\right)^{r-w}\right]^{l-1}, \quad \text{(1d)}$$

where $w = \left\lfloor \frac{r}{2} \right\rfloor$. Since $x_\ell^{(3)}$ and $x_\ell^{(4)}$ both depend solely on $x_{\ell-1}^{(1)}$ and $x_{\ell-1}^{(2)}$, then we can substitute the last two equations into the first two equations, and get that

$$\begin{aligned} x_\ell^{(1)} &= \tilde{f}\left(\epsilon, \delta_L, x_{\ell-1}^{(1)}, x_{\ell-2}^{(1)}, x_{\ell-1}^{(2)}, x_{\ell-2}^{(2)}\right), \ell \geq 2 \\ x_\ell^{(2)} &= \tilde{g}\left(\epsilon, \delta_R, x_{\ell-1}^{(1)}, x_{\ell-2}^{(1)}, x_{\ell-1}^{(2)}, x_{\ell-2}^{(2)}\right), \ell \geq 2 \\ x_1^{(1)} &= x_1^{(2)} = \epsilon,\ x_0^{(1)} = x_0^{(2)} = 1, \end{aligned} \quad (2)$$

for continuous and monotonically non-decreasing functions $\tilde{f}, \tilde{g}$ obtained from (1a)–(1d). By taking the limit $\ell \to \infty$ in (2) and marking $x^{(k)} = \lim_{\ell \to \infty} x_\ell^{(k)}$, $k \in \{1, 2\}$, we get a two-dimensional fixed-point characterization similar to the one derived in [10] for ordinary LDPCL codes

$$\begin{aligned} x^{(1)} &= \tilde{f}\left(\epsilon, \delta_L, x^{(1)}, x^{(1)}, x^{(2)}, x^{(2)}\right) \triangleq f\left(\epsilon, \delta_L, x^{(1)}, x^{(2)}\right), \\ x^{(2)} &= \tilde{g}\left(\epsilon, \delta_R, x^{(1)}, x^{(1)}, x^{(2)}, x^{(2)}\right) \triangleq g\left(\epsilon, \delta_R, x^{(1)}, x^{(2)}\right), \end{aligned} \quad (3)$$

for continuous and monotonically non-decreasing functions f, g.

Remark. *The above derivations refer to the target SB. If one considers helper SBs, one should set δ_L (or δ_R) to 1. In this case, after finding the fixed point of* (3)*, we calculate $\delta_O = 1 - \left(1 - \epsilon \left(\frac{x^{(1)}}{\epsilon}\right)^{\frac{l-1}{l-2}}\right)^{\left\lfloor \frac{r}{2} \right\rfloor}$.*

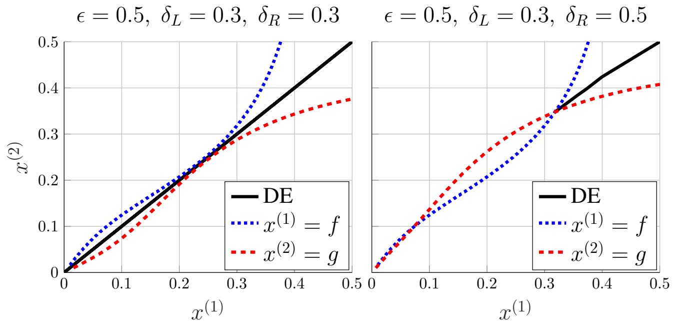

Fig. 4. A graphical representation of the SG DE equations in (3) for the $(l = 3, r = 6, t = 1)$ SC-LDPCL protograph over the $BEC(\epsilon = 0.5)$. In the left plot $\delta_L = 0.3$, $\delta_R = 0.3$, and in the right plot $\delta_L = 0.3, \delta_R = 0.5$.

Example 5. *Figure 4 shows the SG DE equations of the $(3, 6, 1)$ SC-LDPCL protograph over the $BEC(\epsilon = 0.5)$. The incoming erasure messages in the left plot are $\delta_L = 0.3$, $\delta_R = 0.3$, where the DE curve converges to the origin, indicating a decoding success. In the right hand plot we have $\delta_L = 0.3, \delta_R = 0.5$, which leads to a halt in the BP process.*

4. Performance Analysis

In this section we analyze the performance of the semi-global decoder for the family of (l, r, t) SC-LDPCL codes over the BEC. First, we wish to compare the global and SG modes in terms of thresholds and complexity. Evidently, the threshold and complexity induced by the SG mode depend on the number of helpers d: the larger d is, the higher the threshold and complexity are.

Remark. *For simplicity, we assume that $t + 1$ divides r so that $w \triangleq \left\lfloor \frac{r}{t+1} \right\rfloor = \frac{r}{t+1}$. This assumption means that the SG graph $\mathcal{G}_{SG}$ from Definition 2 is right-left symmetric.*

A. Complexity

We assume a fixed number of BP iterations in any decoded sub-block, and a fixed lifting parameter; hence, to compare complexities, we count the number of edges participating in the entire decoding process. We assume an (l, r, t) SC-LDPCL protograph with M SB, each consisting of r VNs. In what follows we mark by χ_G and χ_{SG} the complexities of global and semi-global decoding, respectively.

From Construction 1, there are Mlr edges in the SC-LDPCL protograph, so the global-decoding complexity is $\chi_G = Mlr$. In view of Definition 2 (also Figure 3), in the helper graph there are $(l-t)r + \sum_{i=1}^{t}(r - iw)$ edges, and in the target graph there are $(l - t)r + 2\sum_{i=1}^{t}(r - iw)$ edges. Since we assumed that $w = \frac{r}{t+1}$, then

$$\begin{aligned}\chi_{SG} &= d\left(lr - \frac{wt(t+1)}{2}\right) + (l+t)r - wt(t+1)\\ &= d\left(lr - \frac{rt}{2}\right) + lr,\end{aligned}$$

and the reduction in complexity is given by

$$1 - \frac{\chi_{SG}}{\chi_G} = 1 - \frac{d(l - \frac{t}{2}) + l}{Ml}. \quad (4)$$

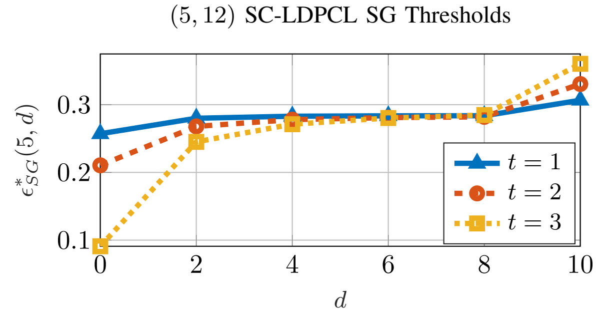

Fig. 5. $\epsilon^*_D(m=5, d)$ for the $(5, 12, t)$ protograph with $M=11$ SBs.

B. Threshold

We define the SG threshold $\epsilon^*_{SG}(m, d)$ as the largest erasure probability $\epsilon \in [0, 1]$ such that SG decoding successfully decodes a target SB $m \in [1 : M]$ using d helper SBs. Figure 5 exemplifies the SG thresholds $\epsilon^*_{SG}(5, d)$ for the $(5, 12, t = 1, 2, 3)$ SC-LDPCL protographs with a total of $M = 11$ SBs. For all $t \in [1 : 3]$, the curves start $(d = 0)$ from the local threshold (see Definition 1), strongly increase with the help of adjacent helpers $(d = 2)$, and end close to the global threshold $(d = M - 1 = 10)$: the global threshold of the $(5, 12, t = 3)$ protograph with $M = 11$ is 0.375, while for $t = 3$ the plots shows $\epsilon^*_{SG}(5, 10) = 0.361$ (3.7% difference). On the other hand, (4) implies that for $t = 3$ the complexity reduction for $d = 10$ equals $1 - \frac{1}{55}\left(10\left(5 - \frac{3}{2}\right) + 5\right) = 27\%$.

C. The Sub-Block Varying BEC

We now examine the SG decoding performance when operating on a channel with SB variability ([11], [12]), i.e., the channel parameter changes between SBs. Let $M \in \mathbb{N}$ be the number of SBs, let $E_1, \ldots, E_M$ be i.i.d. random variables taking values in $[0, 1]$, and let F be the CDF of each E_m. In the SB varying-erasure channel, every SB $m \in [1 : M]$ passes through a $BEC(E_m)$, i.e., first E_m is realized, and then the bits of SB m pass through a $BEC(E_m)$. SG decoding is highly motivated by this practical channel since even if the target SB suffers from high erasure rates, and local decoding fails, potentially the helpers have low erasure rates.

In the following analysis we assume that the protograph is large enough, such that no helper SB is the first or last SB. For every even d and $\underline{\delta}_1, \underline{\delta}_2 \in [0, 1]^t$, we define $p_d(\underline{\delta}_1, \underline{\delta}_2)$ as the SG-decoding success-probability to decode a target SB m with $\frac{d}{2}$ helper SBs to the right (i.e. larger indexes than the target) and $\frac{d}{2}$ helpers to the left, where $\underline{\delta}_1$ and $\underline{\delta}_2$ are the t erasure probabilities incoming from the left to SB $m - \frac{d}{2}$ and from the right to SB $m + \frac{d}{2}$, respectively.

Definition 3. *Let $\underline{\delta}_1, \underline{\delta}_2 \in [0, 1]^t$. We define:*

1) $\epsilon^*(\underline{\delta}_1, \underline{\delta}_2)$*: the target's threshold given that the incoming erasure probabilities are $\underline{\delta}_L = \underline{\delta}_1$, and $\underline{\delta}_R = \underline{\delta}_2$.*
2) $\Delta\colon [0, 1] \times [0, 1]^t \to [0, 1]^t$*: the helper function that calculates the outgoing erasure probabilities $\underline{\delta}_O$ given the SB's erasure probability ϵ and the incoming erasure probabilities $\underline{\delta}_I$, i.e., $\underline{\delta}_O = \Delta(\epsilon, \underline{\delta}_I)$.*
3) $\Delta_k\colon [0, 1]^k \times [0, 1]^t \to [0, 1]^t$*: a recursive function, $k \in \mathbb{N}$*

$$\begin{aligned}&\Delta_0(\underline{\delta}_I) = \underline{\delta}_I,\\ &\Delta_k(\epsilon_1, \ldots, \epsilon_k, \underline{\delta}_I) = \Delta(\epsilon_1, \Delta_{k-1}(\epsilon_2, \ldots, \epsilon_k, \underline{\delta}_I)).\end{aligned}$$

Theorem 1 (proof omitted)**.** *For every varying-erasure channel, and every even $d \geq 0$,*

$$p_d\left(\underline{\delta}_1,\underline{\delta}_2\right)=\mathbb{E}\left[p_{d-2}\left(\Delta\left(E,\underline{\delta}_1\right),\Delta\left(E',\underline{\delta}_2\right)\right)\right],\ d>0,$$
$$p_0(\underline{\delta}_1,\underline{\delta}_2)=\Pr\left(E<\epsilon^*(\underline{\delta}_1,\underline{\delta}_2)\right),$$

where E, E' are i.i.d random variables representing the channel parameter.

Since Theorem 1 involves recursive evaluation of continuous random variables E, E' in $[0,1]$, we cannot use it directly to calculate $p_d\left(\underline{\delta}_1,\underline{\delta}_2\right)$. Thus in the following we quantize the domain to get a provable lower bound.

Theorem 2 (proof omitted)**.** *Let $F(\cdot)$ be the CDF of a SB varying BEC channel, let $K, d \in \mathbb{N}$, and $\underline{\delta}_1,\underline{\delta}_2 \in [0,1]^t$, and let $0=e_0<e_1<e_2<\ldots<e_K=1$ be a partition of $[0,1]$. For every $2d$ indexes $\underline{i}=(i_{-d},\ldots,i_{-1},i_1,\ldots,i_d) \in [1:K]^{2d}$, let*

$$y_{\underline{i}}(\underline{\delta}_1,\underline{\delta}_2) \triangleq \epsilon^*\left(\Delta_d\left(e_{i_{-1}},\ldots,e_{i_{-d}},\underline{\delta}_1\right),\Delta_d\left(e_{i_1},\ldots,e_{i_d},\underline{\delta}_2\right)\right).$$

Then,

$$p_d(\underline{\delta}_1,\underline{\delta}_2) \geq \sum_{\underline{i}\in[1:K]^{2d}} F\left(y_{\underline{i}}(\underline{\delta}_1,\underline{\delta}_2)\right)\prod_{\substack{j=-d\\ j\neq 0}}^{d}\left[F(e_{i_j})-F(e_{i_j-1})\right].$$

Remark. *To get good bounds, it is preferable to have $\epsilon_L \triangleq \epsilon^*(\underline{1},\underline{1})$ and $\epsilon_S \triangleq \epsilon^*(\underline{1},\underline{0})$ as points in the partition $\{e_i\}_{i=1}^K$. For example, one may set*

$$\underline{e}=(0,\epsilon_L,\epsilon_L+\xi_L,\ldots,\epsilon_S,\epsilon_S+\xi_S,\ldots,1), \qquad (5)$$

where $\xi_L=\lfloor\frac{K}{2}\rfloor(\epsilon_S-\epsilon_L)$ and $\xi_S=\left\lfloor\frac{K}{2}\right\rfloor(1-\epsilon_S)$.

In order to ease calculation complexity, we state the next lower bound. Similar to the definition of $p_d(\underline{\delta}_1,\underline{\delta}_2)$, we denote by $\hat{p}_d(\underline{\delta})$ the SG success probability when all d helper SBs have larger indexes than the target, given an input erasure message $\underline{\delta}$.

Proposition 3 (proof omitted)**.** *For every even $d \geq 2$*

$$\begin{aligned}p_d(\underline{1},\underline{1}) \quad &\geq P_L^2 p_{d-2}\left(\underline{0},\underline{0}\right)+2P_L(1-P_L)p_{d-2}\left(\underline{1},\underline{0}\right)\\ &\qquad +\left(1-P_L\right)^2 p_{d-2}\left(\underline{1},\underline{1}\right),\\ p_d(\underline{\delta}_1,\underline{\delta}_2) &\geq P_L+(P_S-P_L)(\hat{p}_{\frac{d}{2}-1}(\underline{\delta}_1)+\hat{p}_{\frac{d}{2}-1}(\underline{\delta}_2))\\ &\qquad +(P_D-2P_S+P_L)\hat{p}_{\frac{d}{2}-1}(\underline{\delta}_1)\hat{p}_{\frac{d}{2}-1}(\underline{\delta}_2),\end{aligned}$$

where $P_L \triangleq \Pr\left(E<\epsilon^*(\underline{1},\underline{1})\right)$, $P_S \triangleq \Pr\left(E<\epsilon^*(\underline{0},\underline{1})\right)$, $P_D \triangleq \Pr\left(E<\epsilon^*(\underline{0},\underline{0})\right)$, and E is the channel-parameter random variable.

Example 6. *Figure 6 compares between the success probability of SG-decoding when all helpers have SB indexes larger than the target ($\hat{p}_d(\underline{1})$), and when half of them have SB indexes smaller than the target ($p_d(\underline{1},\underline{1})$). The plots refer to the ($l=5, r=12, t$) SC-LDPCL protograph over the varying-erasure channel $BEC(E)$, $E \sim U[0,0.4]$. As seen in Figure 6, two-sided SG decoding performs better than one-sided SG decoding for every value of $t=1,2,3$ and every $d \in \{2,4,6,8,10\}$. The bounds are computed according to Theorem 2 and Corollary 3. In Theorem 2 we calculated the bound for $d=2$, with $K=40$ and the partition in* (5).

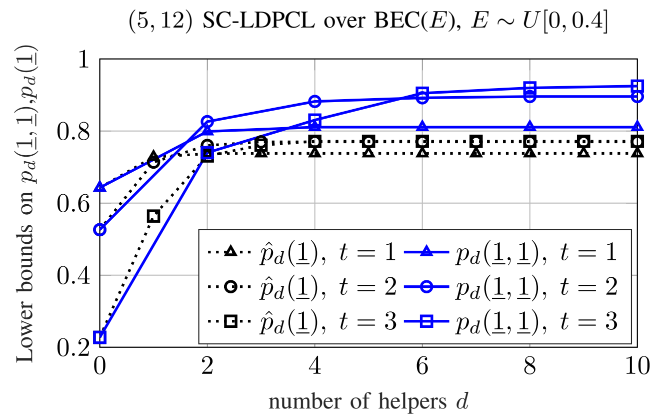

Fig. 6. Lower bounds on $p_d(\underline{1},\underline{1})$ and $\hat{p}_d(\underline{1})$ for the $(5,12,t)$ SC-LDPCL protograph over the $BEC(E)$, $E \sim U[0,0.4]$.

Figure 6 also exemplifies the trade-off between locality and coupling in SC-LDPCL protographs. If $d=0$ (local decoding) it is preferable to use the $t=1$ protograph which is highly localized. However, if $d \geq 6$, the $t=3$ protograph, which is strongly coupled, is superior. In the range $d \in \{2,4\}$ the $t=2$ protograph is superior.

5. Acknowledgment

This work was supported in part by the US-Israel Binational Science Foundation.

References

[1] A. J. Felström and K. S. Zigangirov, "Time-varying periodic convolutional codes with low-density parity-check matrix," *IEEE Trans. on Inf. Theory*, vol. 45, no. 6, pp. 2181–2191, Sep. 1999.

[2] S. Kudekar, T. Richardson and R. L. Urbanke, "Spatially-coupled ensembles universally achieve capacity under belief propagation," *IEEE Trans. on Inf. Theory*, vol. 59, no. 12, pp. 7761–7813, Dec. 2013.

[3] D. G. M. Mitchell, M. Lentmaier and D. J. Costello, "Spatially coupled LDPC codes constructed from protographs," *IEEE Trans. on Inf. Theory*, vol. 61, no. 9, pp. 4866–4889, Sep. 2015.

[4] D. G. M. Mitchell, M. Lentmaier and D. J. Costello, "AWGN channel analysis of terminated LDPC convolutional codes," *Proc. 2011 Inf. Theory and App. Workshop*, La Jolla, CA, 2011, pp. 1-5.

[5] A. E. Pusane, A. J. Felström, A. Sridharan, M. Lentmaier, K. S. Zigangirov and D, J. Costello, "Implementing aspects of LDPC convolutional codes," *IEEE Trans. on Comm.*, vol. 56, no. 7, pp. 1060–1069, July. 2008.

[6] A. R. Iyengar, P. H. Siegel, R. L. Urbanke and J. K. Wolf, "Windowed decoding of spatially coupled codes," *IEEE Trans. on Inf. Theory*, vol. 59, no. 4, pp. 2277–2292, April 2013.

[7] M. Papaleo, A. R. Iyengar, P. H. Siegel, J. K. Wolf, G. E. Corazza, "Windowed erasure decoding of LDPC convolutional codes," *Proc. 2010 IEEE Inf. Theory Workshop*, pp. 78-82, Cairo, Egypt, Jan. 2010.

[8] M. Lentmaier, M. M. Prenda and G. P. Fettweis, "Efficient message passing scheduling for terminated LDPC convolutional codes," *Proc. 2011 IEEE Int. Symp. on Inf. Theory*, pp. 1826-1830, St. Petersburg, Russia, Aug. 2011.

[9] Y. Cassuto, E. Hemo, S. Puchinger and M. Bossert, "Multi-block interleaved codes for local and global read access," *Proc. 2017 IEEE Int. Symp. on Inf. Theory*, pp. 1758–1762, Aachen, Germany, June 2017.

[10] E. Ram and Y. Cassuto, "LDPC codes with local and global decoding," *Proc. 2018 IEEE Int. Symp. on Inf. Theory*, pp. 1151–1155, Vail, Colorado, USA, June 2018.

[11] E. Ram and Y. Cassuto, "Spatially coupled LDPC codes with random access," *IEEE 10th Int. Symp. on Turbo Codes & Iterative Inf. Processing*, Hong-Kong, Dec. 2018.

[12] R. J. McEliece and W. E. Stark, "Channels with block interface," *IEEE Trans. on Inf. Theory*, vol. 30, no. 1, pp. 44–53, Jan. 1984.