Spatially Coupled LDPC Codes with Random Access

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***Abstract*—A new type of spatially-coupled (SC) LDPC codes motivated by practical storage applications is presented. SC-LDPCL codes (suffix 'L' stands for locality) can be decoded locally at the level of sub-blocks that are much smaller than the full code block, thus providing random access to the coded information. The same code can also be decoded globally using the entire code block (as usual), for increased data reliability. The main contribution of the paper is a construction of SC-LDPCL codes that allows controlling the trade-off between local and global correction performance.**

1. Introduction

[1] One of the most promising error correcting schemes is the family of SC-LDPC codes [2]. In [3] it was proven that SC-LDPC codes achieve capacity universally on memoryless binary symmetric channels due to a phenomenon called *threshold saturation*. Moreover, their special structure where bits participating in a particular parity-check equation are spatially close to each other renders a locality property that can be exploited to implement low-latency high-throughput belief-propagation based decoders; such decoders are pipelined decoders [2], [8] and window decoders [7], [4].

However, to benefit from the threshold-saturation effect, very long block lengths are needed. This is a major drawback in the context of data-storage applications with moderate size read units. In this paper, we enhance the functionality of SC-LDPC codes by designing protographs (see [9], [6]) equipped with a strong locality structure that enables decoding sub-blocks in a random access fashion, thereby only requiring to read fragment of the entire codeword; we call these codes SC-LDPCL codes. This sub-block random access property makes the SC-LDPCL code a multi-block coding scheme where codewords are divided into a number of sub-blocks; each sub-block is a codeword of one code, and can be decoded independently from other sub-blocks (local decoding), and the concatenation of the sub-blocks forms a codeword of another (stronger) code which is decoded if local decoding failed (global decoding). In [10] we proposed a multi-block scheme based on ordinary LDPC codes, presenting an optimal capacity-achieving construction, and in [1] a Reed-Solomon-based multi-block scheme was proposed. Our results include bounds on the thresholds of protototographs, yielding a negative result on random access in existing SC-LDPC codes. Then we give a construction of SC-LDPCL codes and illustrate the trade-off between local and global decoding thresholds. In addition, we suggest a decoding strategy we call *semi-global decoding* that is highly motivated by the random access ability of SC-LDPCL codes and by practical data-storage models, in which variability is introduced to the channel quality.

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2. Preliminaries

A. Protograph Based LDPC Codes

An LDPC protograph is a bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{C}, \mathcal{E})$, where $\mathcal{V}, \mathcal{C}$ and $\mathcal{E}$ are the sets of variable nodes (VNs), check nodes (CNs) and edges, repsectively; parallel edges are allowed. For every VN $v \in \mathcal{V}$, d_v denotes its edge degree. Similarly, d_c denotes the edge degree of CN $c \in \mathcal{C}$. As in ordinary LDPC codes, a protograph is said to be (l, r)-regular if for every $v \in \mathcal{V}$, $d_v = l$, and for every $c \in \mathcal{C}$, $d_c = r$. A Tanner graph, representing a protograph-based LDPC code, is generated from a protograph $\mathcal{G}$ by a lifting ("copy-and-permute") operation specified by a lifting parameter L (see [9]). The design rate corresponding to the derived Tanner graph is independent of L and given by $R_{\mathcal{G}} = 1 - \frac{|\mathcal{C}|}{|\mathcal{V}|}$. As $L \to \infty$, one can analyze the performance of the BP decoder on the resulting Tanner graph via density evolution on the original protograph. Formally, for the binary erasure channel (BEC) we have:

Fact 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{C}, \mathcal{E})$ be an LDPC protograph, let $v \in \mathcal{V}$ be a variable node of degree d_v, and let $c \in \mathcal{C}$ be a check node of degree d_c. Let $\{e_1^v, e_2^v, \ldots, e_{d_v}^v\}$ be the set of all edges connected to v, and let $\{e_1^c, e_2^c, \ldots, e_{d_c}^c\}$ be the set of all edges connected to c. Let $\epsilon \in [0, 1]$, and consider a transmission over the $BEC(\epsilon)$, of a codeword from a binary linear code that corresponds to a Tanner graph lifted from $\mathcal{G}$ with a lifting parameter L, denoted by $\mathcal{G}^L$. For every $t \in \{1, 2, \ldots, d_v\}$, let $x_l(e_t^v)$ and $u_l(e_t^v)$ be the fraction of e_t^v-type edges that carry VN-to-CN and CN-to-VN erasure messages, respectively, after l BP iterations on $\mathcal{G}^L$. Similarly, for every $s \in \{1, 2, \ldots, d_c\}$, let $x_l(e_s^c)$ and $u_l(e_s^c)$ be the fraction of e_s^c-type edges that carry VN-to-CN and CN-to-VN erasure messages after l BP iterations on $\mathcal{G}^L$. Then, as $L \to \infty$*

$$x_l\left(e_t^v\right) = \epsilon \cdot \prod_{\substack{1 \le t' \le d_v \\ t' \ne t}} u_l\left(e_{t'}^v\right), \tag{1a}$$

$$u_l\left(e_s^c\right) = 1 - \prod_{\substack{1\le s'\le d_c\\ s'\neq s}} \left(1 - x_{l-1}\left(e_{s'}^c\right)\right), \tag{1b}$$

$$x_{-1}\left(e_t^v\right) = 1. \tag{1c}$$

Moreover, as $L \to \infty$ the probability that v is erased after l BP iterations is given by

$$P_{e,l}(v,\epsilon) = \epsilon \prod_{1\le t\le d_v} u_l\left(e_t^v\right). \tag{2}$$

The BP decoding threshold of $\mathcal{G}$ is defined by

$$\epsilon_{BP}^*\left(\mathcal{G}\right) = \sup\{\epsilon \in [0,1] \colon \max_{v\in\mathcal{V}} \lim_{l\to\infty} P_{e,l}(v,\epsilon) = 0\}. \tag{3}$$

For simplicity of notations, we remove the subscript BP from the threshold notation for the rest of the paper. If the MAP threshold is discussed, then we add a subscript MAP, i.e., write ϵ_{MAP}^*.

A protograph $\mathcal{G} = (\mathcal{V}, \mathcal{C}, \mathcal{E})$ is frequently represented through a bi-adjacency matrix $H_{\mathcal{G}}$, where the VNs in $\mathcal{V}$ are indexed by the columns of $H_{\mathcal{G}}$, and the CNs in $\mathcal{C}$ by the rows. Note that since parallel edges are permitted, $H_{\mathcal{G}}$ can be non-binary although the code generated by it is, in fact, a binary code. In this matrix representation, we write $\epsilon^*\left(H_{\mathcal{G}}\right)$ to denote the decoding threshold defined in (3).

B. SC-LDPC Codes

A SC-LDPC (l,r)-regular protograph is constructed by coupling together (l,r)-regular protographs and truncating the resulting chain. This coupling operation introduces a convolutional structure to the code, and we visualize it through the matrix representation of the protograph. Let $\mathbf{1}^{l\times r}$ be an all-ones matrix with l rows and r columns representing an (l,r)-regular protograph with r VNs and l CNs. In the remainder of the paper we focus on spatial coupling with two matrices $B_1, B_2 \in \{0,1\}^{l\times r}$ such that $B_1, B_2 \neq 0$ and $B_1 + B_2 = \mathbf{1}^{l\times r}$, but the results readily extend to coupling more matrices. Coupling a limitless number of copies of $\mathbf{1}^{l\times r}$ consist of diagonally placing copies of $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ as in Figure 1 (a). Note that different choices of B_1 yield different protographs. These choices are known as *edge spreading rules* and introduce a degree of freedom to the design (see [6] and references therein). By truncating the infinite matrix in Figure 1 (a) at some width, and removing all-zero rows, an "(l,r)-regular" spatially-coupled LDPC protograph is constructed. This truncation results in a small number of terminating CNs (of low degree), which cause a decrease in design rate and an increase in the decoding threshold, compared to the underlying (l,r)-regular code ensemble. However, as the width of the coupled chain increases, the design rate of the coupled protograph converges to the design rate of the underlying (l,r)-regular code ensemble, and its BP threshold exhibits a phenomenon known as *threshold saturation*, where it converges to the MAP threshold of the underlying (l,r)-regular code ensemble (see [3]). In other words, if the protograph is large enough, then the BP decoder suffers a negligible performance loss compared to the optimal MAP decoder for the underlying regular code. Threshold saturation of spatially-coupled LDPC code ensembles enables a construction of capacity-achieving regular sequences (i.e., by increasing l and r while keeping $\frac{l}{r}$ fixed), and can be explained through a "reliability wave" argument: even if the channel parameter is in the range $(\epsilon_{BP}^*(l,r), \epsilon_{MAP}^*(l,r))$, the low-degree CNs at the endpoints of the coupled protograph carry reliable messages, and as iterations progress, this reliability propagates through the protograph to the center, resolving all of the VNs.

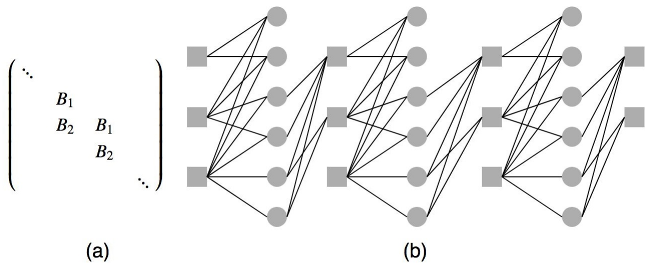


Fig. 1. (a) matrix representation of a coupled protograph; (b) a coupled protograph of the $(3,6)$-regular SC code ensemble.

Example 1. *Figure 1 (b) illustrates the spatially-coupled $(3,6)$ protograph generated by*

$$B_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{4}$$

and truncated at width of 18 *columns (VNs). The design rate of the coupled protograph is* 0.389*, and the BP thresholds is* $0.512 > \epsilon^*(3,6) = 0.4294$*. Moreover, as the chain width tends to infinity, the design rate of the coupled protograph tends to $\frac{1}{2}$, and the BP threshold tends to* $0.4882 = \epsilon_{MAP}^*(3,6)$*. Finally, we can approach capacity by using higher degrees, e.g., if $(l,r) = (5,10)$, then* $\epsilon_{MAP}^*(5,10) = 0.4995$.

3. Multi-Block SC-LDPC Codes

Consider a coupled protograph $\mathcal{G} = (\mathcal{V}, \mathcal{C}, \mathcal{E})$. To obtain a multi-block LDPC code (as done in [10] without spatial coupling), we divide $\mathcal{V}$ into $M > 1$ disjoint sets $\{\mathcal{V}_m\}_{m=1}^M$, and refer to $\mathcal{V}$ as the code-block and to the M subsets $\{\mathcal{V}_m\}_{m=1}^M$ as sub-blocks (SBs). In what follows, let $H = H_{\mathcal{G}}$ be a bi-adjacency matrix representing the coupled protograph $\mathcal{G}$, and let $m \in \{1, 2, \ldots, M\}$ be an SB index.

Definition 1.

1) *If VN $j \in \mathcal{V}$ belongs to SB m, we write $j \in \mathcal{V}_m$.*
2) *CN $i \in \mathcal{C}$ is said to be a local check (LC) in SB m if and only if*

$$\{j \colon H_{i,j} = 1\} \subseteq \mathcal{V}_m,$$

and we write $i \in \mathcal{C}_m$.

3) *The local protograph of SB m is the sub-graph $\mathcal{G}_m = (\mathcal{V}_m, \mathcal{C}_m, \mathcal{E}_m)$, where $\mathcal{E}_m$ is the set of edges in $\mathcal{E}$ that connect between $\mathcal{V}_m$ and $\mathcal{C}_m$.*

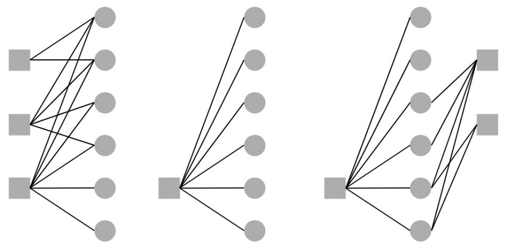


Fig. 2. The *local protographs* from Example 2: (a) protograph $\mathcal{G}_1$; (b) protograph $\mathcal{G}_2$; (c) protograph $\mathcal{G}_3$.

4) *The global and local BP decoding thresholds are given by $\epsilon^*_G \triangleq \epsilon^*(\mathcal{G})$ and $\epsilon^*_m \triangleq \epsilon^*(\mathcal{G}_m)$, respectively.*

Example 2. *Let $\mathcal{G}$ be the coupled protograph from Example 1 (see Figure 1 (b)). If we divide $\mathcal{V}$ into $M = 3$ equally sized SBs, then $\mathcal{V}_1 = \{1,2,3,4,5,6\}$, $\mathcal{V}_2 = \{7,8,9,10,11,12\}$, $\mathcal{V}_3 = \{13,14,15,16,17,18\}$, and $\mathcal{C}_1 = \{1,2,3\}$, $\mathcal{C}_2 = \{6\}$, $\mathcal{C}_3 = \{9,10,11\}$. The local protographs $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ are illustrated in Figure 2. The local decoding thresholds in this case are all zero, i.e., $\epsilon^*_1 = \epsilon^*_2 = \epsilon^*_3 = 0$. As we will see later, zero local thresholds are a general phenomenon in SC-LDPC codes, unless proper design measures are taken.*

A. Basic Results

Giving an explicit expression for the threshold of protographs is, in general, not an easy task since many densities should be tracked. In what follows, we give bounds on the threshold.

Lemma 1. *Let H be a bi-adjacency matrix representing an LDPC protograph $\mathcal{G} = (\mathcal{V}, \mathcal{C}, \mathcal{E})$. Let $\mathcal{J} \subseteq \{1,2,\ldots,|\mathcal{V}|\}$ and $\mathcal{I} \subseteq \{1,2,\ldots,|\mathcal{C}|\}$ be a set of column and row indices, respectively, and let $H_{\mathcal{J}}$ (resp. $H^{(\mathcal{I})}$) be the sub-matrix consisting of the columns (resp. rows) of H indexed by $\mathcal{J}$ (resp. $\mathcal{I}$). Then,*

$$\epsilon^*\left(H^{(\mathcal{I})}\right) \le \epsilon^*(H) \le \epsilon^*(H_{\mathcal{J}}). \quad (5)$$

Corollary 2. *Let $\epsilon^*(l,r)$ denote the BP decoding threshold of (l,r)-regular LDPC codes, i.e., $\epsilon^*(l,r) = \inf_{(0,1]} \frac{x}{(1-(1-x)^{r-1})^{l-1}}$. If H has a sub-matrix $H_{\mathcal{J}}$ as in Lemma 1 which is (l_1,r_1)-regular, then $\epsilon^*(H) \le \epsilon^*(l_1,r_1)$. If H has a sub-matrix $H^{(\mathcal{I})}$ as in Lemma 1 which is (l_2,r_2)-regular, then $\epsilon^*(H) \ge \epsilon^*(l_2,r_2)$.*

Lemma 3. *Consider the LDPC protograph represented by the bi-adjacency matrix $H = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Then, $\epsilon^*(H) = 0$.*

Proof. For every iteration l, let $p_l(i,j)$ and $q_l(i,j)$ be the probabilities that the edge between CN (row) i and VN (column) j carries an erasure VN-to-CN and CN-to-VN messages, respectively, and let $\epsilon \in (0,1)$. In view of (1a), for every iteration l, $p_l(2,3) = \epsilon$, which implies due to (1b), that

$$q_l(2,1), q_l(2,2) > 1 - (1-\epsilon) = \epsilon > 0, \quad \forall l \ge 0. \quad (6)$$

Subsequently, it follows from (1a) that for every iteration l, $p_l(1,2), p_l(1,1) > 0$, so $q_l(1,1), q_l(1,2) > 0$, which, in view of (2) and (6), implies that for every iteration l, $P_{e,l}(1,\epsilon), P_{e,l}(2,\epsilon) > 0$. Since this is true for every $\epsilon > 0$, we get $\epsilon^*(H) = 0$. □

Theorem 4. *Let H be a bi-adjacency matrix representing an SC-LDPC protograph $\mathcal{G} = (\mathcal{V}, \mathcal{C}, \mathcal{E})$ constructed by truncating the infinite matrix in Figure 1 (a), and suppose $\mathcal{V}$ is divided into $M > 1$ SBs. If there are no 2 rows in B_1 nor in B_2 that are all ones, then*

$$\epsilon^*_m = 0, \quad 1 \le m \le M.$$

Proof (sketch). Let $m \in \{1,2,\ldots,M\}$ be an SB index and let H_m be the bi-adjacency matrix representing the local protograph of SB m. From the construction of SC-LDPC protographs in Figure 1 (a), and since there are no 2 rows in B_1 nor in B_2 that are all-one rows, one can always find a sub-matrix of H_m denoted by $\tilde{H}_m$ consisting of some 3 columns of H_m, such that at least one of the following holds: 1) $\tilde{H}_m$ has a zero column, or 2) $\tilde{H}_m$ is as in Lemma 3, or 3) $\tilde{H}_m$ has a column sub-matrix that is $(l=1,r)$-regular. In all 3 cases, $\epsilon^*\left(\tilde{H}_m\right) = 0$, so in view of Lemma 1, $\epsilon^*(H_m) = 0$. □

Corollary 5. *If $l = 2$, then no (l,r)-regular SC-LDPC protograph can have a non-zero local decoding threshold.*

Remark. *The family of protographs covered by Theorem 4 is larger than it may seem in a first look. For example, the (l,r) SC-LDPC-CC ensemble from [6, Definition 3] with $l = \gcd(l,r)$, is included in Theorem 4.*

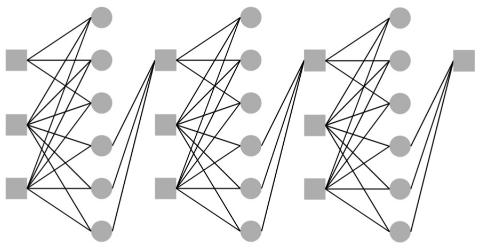
B. A SC-LDPCL construction

Motivated by Theorem 4, we introduce a construction of (l,r)-regular SC-LDPC protographs that enable local decoding. We assume that for every $m \in \{1,2,\ldots,M\}$, $|\mathcal{V}_m| = r$ (this can be readily generalized to lengths divisible by r). In addition, in view of Corollary 5, we assume that $l \ge 3$. The inputs to the construction are: the degrees (l,r), the number of SBs M, and a coupling parameter $t \in \{1,2,\ldots,l-2\}$; we call the resulting protograph an (l,r,t) spatially-coupled protograph with M SBs. As we will see, t serves as a design tool to control the trade-off between local and global decoding thresholds.

Construction 1 (SC-LDPCL)**.** *Let A_1 be a $t \times r$ matrix given by*

$$A_1 = \begin{pmatrix} \underline{1} & & & & \underline{0} \\ \underline{1} & \underline{1} & & \mathit{0} & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} & & \underline{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{1} & \underline{1} & \underline{1} & \cdots \;\; \underline{1} & \underline{0} \end{pmatrix}, \quad (7)$$

where $\underline{1}$ and $\underline{0}$ are length-$\lfloor \frac{r}{t+1} \rfloor$ all-one vector and length-$\left(r - t\lfloor \frac{r}{t+1} \rfloor\right)$ all-zero vector, respectively. Let A_2 be an all-ones $(l-t) \times r$ matrix. We build the (l,r,t) protograph as in



(b)

Fig. 3. The $(3,6,1)$ SC-LDPCL protograph with $M=3$ SBs.

Figure 1 (a) with M copies of $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ *on the diagonal, where* $B_1 = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ *and* $B_2 = \mathbf{1}^{l\times r} - B_1$.

The resulting coupled protograph $\mathcal{G}$ has rM VNs and $lM+t$ CNs, so the design rate is $R_{\mathcal{G}} = 1 - \frac{l}{r} - \frac{t}{rM}$. For every $m \in \{2,\ldots,M-1\}$, the local graph $\mathcal{G}_m$ is represented by A_2 which is $(l-t,r)$-regular, and for $m=1$ and $m=M$, the local graph $\mathcal{G}_m$ is represented by $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $\begin{bmatrix} A_2 \\ \bar{A}_1 \end{bmatrix}$, respectively, where $\bar{A}_1$ is the complement of A_1. Thus, for every $m \in \{2,\ldots,M-1\}$, $\epsilon^*_m = \epsilon^*(l-t,r) > 0$, and Lemma 1 implies that for $m \in \{1,M\}$, $\epsilon^*_m \geq \epsilon^*(l-t,r) > 0$, where $\epsilon^*(l-t,r) > 0$ is the BP threshold of the $(l-t,r)$-regular LDPC code ensemble.

Example 3. *Figure 3 illustrates the $(3,6,1)$ SC-LDPCL protograph with $M=3$ SBs. In this case $A_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$, the design rate is $R=0.4444$ and the thresholds are: $\epsilon^*_{\mathcal{G}} = 0.4772$, $\epsilon^*_1 = \epsilon^*_3 = 0.4298$, and $\epsilon^*_2 = 0.2$. Note that ϵ^*_2 corresponds to the $(2,6)$-regular LDPC code ensemble.*

Example 4. *Table I details the design rate and thresholds of the $(5,10,t)$ SC-LDPCL protograph for $t=1,2,3$, with $M=6$ SBs.*

TABLE I

t	ϵ^*_1	$\epsilon^*_2,\ldots,\epsilon^*_5$	ϵ^*_6	$\epsilon^*_{\mathcal{G}}$	R
1	0.3667	0.3079	0.3667	0.3734	0.4883
2	0.4017	0.2538	0.3935	0.4148	0.4667
3	0.3333	0.1111	0.4263	0.4654	0.45

Remarks.

1) *The coupling parameter t serves as a design tool that controls the trade-off between the local and global thresholds. When $t=1$, more CNs are LCs, and the local threshold and design rate are high. On the other end, when $t=l-1$, the local threshold is 0 and the protograph is strongly coupled; this case corresponds to a standard SC-LDPC protograph (i.e., no locality); that is, Construction 1 can be viewed as a generalization of the SC-LDPC construction.*
2) *The specific choice of A_1 in (7) is motivated by the "reliability wave" argument (see Section 2). It is reasonable to induce this "reliability wave" with a uniform increase in the CN degrees, and this is exactly what we have done in Construction 1.*

4. Semi-Global Decoding

In this section, we suggest a decoding strategy called *semi-global* decoding, in which the decoder decodes a target SB $m \in \{1,2,\ldots,M\}$ with the help of additional d neighbor SBs $\{m+1,m+2,\ldots,m+d\}$ (assuming $m+d \leq M$) in a sequential fashion. d is a parameter that limits the worst-case number of SBs read for decoding one SB; hence, the smaller d is, the better random access the code gives for single SBs. In the first step of the semi-global decoding mode the decoder tries to decode SB m, and returns the extracted information upon success. If it fails, it tries to decode SB $m+1$, and upon success uses the decoded information to decode SB m again. If it fails on SB $m+1$ it tries SB $m+2$ and backtracks to SB m. This procedure continues upto SB $m+d$.

The semi-global decoding mode is highly motivated by the locality property of sub-blocks in SC-LDPCL codes (SBs can be decoded locally), the spatial coupling of SBs (SBs can help their neighbor SBs), and by practical channels in storage devices, i.e., channels with variability. In particular, we consider a channel model originally proposed in [5], in which the channel parameter changes between sub-blocks.

Remark. *Since the SBs at the endpoints of the SC-LDPCL protograph (i.e., SB 1 and M) are the most reliable SBs, the* semi-global *decoding strategy described above should be modified such that if $m-d \leq 1$ and $m+d < M$, then SBs $\{m-1,m-2,\ldots,\max\{m-d,1\}\}$ participate in the decoding procedure instead of SBs $\{m+1,m+2,\ldots,m+d\}$. However, in the following analysis we will assume for simplicity that*

$$m+d < M, \quad \textit{and} \quad m-d > 1, \tag{8}$$

which implies that there is no difference between these 2 options (i.e., $+$ or $-$).

A. Channel Model

Let $E_1, E_2, \ldots, E_M$ be i.i.d random variables taking values in $[0,1]$, and let F be the CDF of E_m, i.e., for every $m \in \{1,2,\ldots,M\}$ and $\epsilon \in \mathbb{R}$, $\Pr(E_m \leq \epsilon) = F(\epsilon)$, where $F(0)=0$ and $F(1)=1$. In the channel introduced in [5], all of the bits of SB m are transmitted over the channel $BEC(E_m)$; in other words, first E_m is realized, and then the bits of SB m pass through the same channel $BEC(E_m)$. Note that in our case, each SB is of length rL where r is the CN degree and L is the lifting parameter.

B. Analysis

Definition 2 (Semi-Global Threshold)**.** *Let $\mathcal{G} = (\mathcal{V},\mathcal{C},\mathcal{E})$ be an SC-LDPCL protograph constructed by Construction 1, let H be its corresponding bi-adjacency matrix, and let $m \in \{1,2,\ldots,M\}$ be an SB index.*

1) *If $i \in \mathcal{C}$ is not a local check, call it a coupling check (CC), and if in addition*

$$\{j\colon H_{i,j}=1\} \subseteq \mathcal{V}_m \cup \mathcal{V}_{m+1},$$

we say that CN i couples SB m and SB $m+1$, and write $i \in \mathcal{C}_m^+$.

2) *The extended local protograph of SB m is the subgraph $\mathcal{G}_m^+ = (\mathcal{V}_m, \mathcal{C}_m \cup \mathcal{C}_m^+, \mathcal{E}_m \cup \mathcal{E}_m^+)$, where $\mathcal{E}_m^+$ is the set of edges in $\mathcal{E}$ that connect $\mathcal{V}_m$ and $\mathcal{C}_m^+$.*
3) *The semi-global BP decoding threshold of SB m is given by $\epsilon_{S,m}^* \triangleq \epsilon^*\left(\mathcal{G}_m^+\right)$.*

In view of the left inequality of (5) and the fact that $\mathcal{G}_m \subseteq \mathcal{G}_m^+$, but $\mathcal{G}_m^+$ has more CNs (rows) than $\mathcal{G}_m$, we have

$$\epsilon_{S,m}^* \geq \epsilon_m^*, \quad \forall m \in \{1, 2, \ldots, M\}, \tag{9}$$

i.e., decoding SB $m+1$ increases the reliability of SB m; in many cases, the semi-global threshold $\epsilon_{S,m}^*$ is much closer to the global threshold ϵ_G^* than to the local threshold ϵ_m^*.

Example 5. *Consider the SC-LDPCL protographs from Example 4, and let $m = 3$. For $t = 1, 2, 3$, we have the three respective semi-global thresholds $\epsilon_{S,3}^* = 0.3667, 0.3935, 0.4263$. For $t = 2$ The bi-adjacency matrix corresponding to the extended local protograph of SB $m = 3$ is given by*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Remarks.

1) *In Item 1) of Definition 2 we consider CNs coupling only immediately adjacent SBs. This could be generalized to arbitrary SBs, but for Construction 1 this definition is sufficient. Furthermore, the definition considers CNs connecting SB m and $m+1$, but one can consider CCs connecting SB m and $m-1$ as well.*
2) *In view of the assumption in (8), we can assume that all SBs participating in the semi-global decoding mode have the same local and semi-global threshold, i.e., for every $k \in \{0, 1, \ldots, d\}$, $\epsilon_{m+k}^* = \epsilon_L^*$, $\epsilon_{S,m+k}^* = \epsilon_S^*$.*

The channel model described in Section 4-A induces a success probability for local and semi-global modes in the asymptotic region (where the lifting parameter $L \to \infty$). For example, as $L \to \infty$, local decoding SB m succeeds if and only if $E_m < \epsilon_L^*$. Thus, $\Pr(\text{local success}) = F(\epsilon_L^*)$ (assuming continuity of F at ϵ_L^*). In view of this connection, we define the quantities $P_L \triangleq F(\epsilon_L^*)$ and $P_S \triangleq F(\epsilon_S^*)$ (or $\lim_{\epsilon \uparrow \epsilon_L^*} F(\epsilon)$ and $\lim_{\epsilon \uparrow \epsilon_S^*} F(\epsilon)$ for the non-continuous cases).

Proposition 6. *For every $d \geq 0$, let $\mathcal{D}_d$ be the event of semi-global decoding success, as $L \to \infty$, when using a length-d decoding chain of SBs. Then,*

$$P_L \frac{1 - q^{d+1}}{1 - q} \leq \Pr(\mathcal{D}_d) \leq P_s \tag{10}$$

where $q \triangleq P_S - P_L$.

Example 6. *Figure 4 exemplifies the lower bound in* (10) *for the $(l = 5, r = 10, t)$ SC-LDPCL protographs with $t = 1, 2, 3$ from Examples 4 and 5 over the erasure channel with sub-block variability, where $E_m \sim U[0, 0.4]$. As seen in the plot, the best choice of t depends on the parameter d: when $t = 1$, the resulting protograph is highly localized (high local threshold and low semi-global threshold), so it is superior for short decoding chains $d = 0, 1$; when $t = 3$, the protograph is strongly coupled (low local threshold and high semi-global threshold, so it is superior for long decoding chains $d > 10$; the $t = 2$ protograph is superior for moderate decoding chains $2 \leq d \leq 10$. Finally, for every $t \in \{1, 2, 3\}$ the lower bound tends to the value $\frac{P_L}{1-q} = \frac{P_L}{1+P_L-P_S}$ as $d \to \infty$, and this convergence is exponentially fast.*

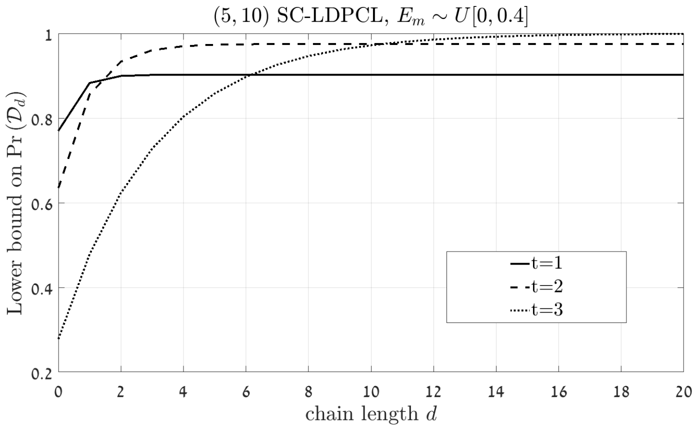


Fig. 4. The lower bound from Proposition 6 as a function of the chain length d, for the $(l = 5, r = 10, t)$ SC-LDPCL protographs with $t = 1, 2, 3$, over the sub-block varying BEC with $E_m \sim U[0, 0.4]$.

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