Low-Delay Erasure-Correcting Codes with Optimal Average Delay

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***Abstract*—The objective of low-delay codes is to protect communication streams from erasure bursts by minimizing the time between the packet erasure and its reconstruction. Previous work has concentrated on the constant-delay scenario, where all erased packets need to exhibit the same decoding delay. We consider the case of heterogeneous delay, where the objective is to minimize the average delay across the erased packets in a burst. We derive delay lower bounds for the average case, and show that they match the constant-delay bounds only at a single rate point 0.5. We then construct codes with optimal average delays for the entire range of code rates. The construction for rates under 0.5 achieves optimality for every erasure instance, while the construction for rates above 0.5 is optimal for an infinite number, but not all, of the erasure instances.**

I. Introduction

There are many practical scenarios where a communication system needs to reconstruct corrupt or lost data with minimal delay. These scenarios are common in communication devices with small buffers, and in systems interacting with the physical world. When delay becomes a major concern, one needs to explicitly introduce it to the coding model. A very elegant coding model involving decoding delay has been introduced by Martinian [1], which in particular showed that MDS codes, a common panacea for erasures, are *not* optimal when burst-erasure correction is needed with low delay. The paradigm developed by Martinian – its constructions and bounds – was the basis for several follow-up works promoting different scenarios of low-delay communication: [2] (flexible construction), [3] (multiple bursts), and [4] (multi-user).

The prior work has concentrated on the case where every packet in the stream needs to exhibit the same delay. There are many practical scenarios where this restriction is not necessary. For example, in many control networks (e.g. automotive networks), the nodes not only forward data, but also perform computations on it. In such networks it is preferable to obtain part of the data very early, and start the computation while additional packets are being reconstructed. For such scenarios we are considering in this paper *heterogeneous delay*, and seek to minimize the *average delay* of reconstruction, calculated over the packets erased in a burst-erasure instance.

As it turns out, there is a big gap between the achievable delays in the average and constant regimes. It is possible to reduce the delay considerably if one lifts the constant requirement. In particular, in Section III we derive bounds for the average case, and show that they only match the constant case at a single rate point $R = 0.5$. We then move in Section IV to construct codes that match the average-delay bounds for the entire range of code rates. One construction for rates $R \geq 0.5$ achieves optimality for the infinite number of burst instances characterized by their phase with respect to the construction. Another construction for rates $R < 0.5$ achieves optimality for every burst instance.

II. Low Delay Codes

A. Previous work – constant delay

The low-delay coding paradigm was founded in the work of Martinian [1]. We now briefly review its model and main results. A low-delay erasure coding scheme facilitates packet transmission subject to bursts of erasures. Each packet comprises a number of symbols taken from some alphabet. The objective of the low-delay code is to reconstruct the symbols of an erased packet *at a minimum temporal delay* from the time of the packet's original transmission. Hence low-delay codes are characterized by a tradeoff between three parameters: the erasure-burst length B, the reconstruction delay T – both B and T are measured in units of packets – and the code rate R defined as the ratio between the information content of the packet and its total size. These three parameters are shown in [1] to satisfy the following inequality relation

$$T \geq \max\left\{B\ ,\ B \cdot \frac{R}{1-R}\right\}. \tag{1}$$

The implication of (1) is a lower bound on the reconstruction delay given the burst length and the code rate. Another important result of [1] is that this bound is achievable, through an explicit family of codes whose parameters satisfy (1) with equality.

The above model and the codes constructed for it have the property of *constant delay*, i.e, each erased packet in a burst is reconstructed with the same delay from the time of its transmission. Our objective in this paper is to lift the constant-delay property, and examine codes that provide different delays to different packets in the burst.

B. Heterogeneous delay

To deal with heterogeneous delay, we define T_i to be the delay of the i-th erased packet in a given erasure burst. That is, T_i packets were transmitted between the transmission of packet i and its eventual reconstruction. T_1 is the delay of the first erased packet in the burst, i.e, the oldest one, and T_B is the last, most recently erased packet. This is shown in Figure 1.

A natural performance measure for heterogeneous delay is the average delay across the erasure burst.

Definition 1 *The average delay over all B erased packets is defined as*

$$\bar{T} = \frac{\sum_{i=1}^{B} T_i}{B}. \tag{2}$$

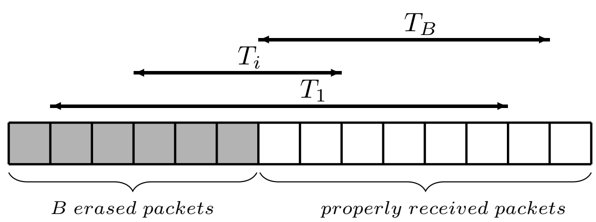

Figure 1: The non-constant delay T_i.

When using a constant-delay code, then clearly $\bar{T} = T$, so the average delay must obey the lower bound (1). However, for the heterogeneous-delay case (1) in general is *not* a lower bound on $\bar{T}$. It is therefore interesting to investigate the possibility to improve the average delay beyond the best-achievable constant-delay codes. This investigation will include both lower bounds on the average delay, and code constructions with lower average delays. Toward that objective the following definition will be useful.

Definition 2 *The recovery delay of the i-th packet, denoted κ_i, is the number of packets that were received from the end of the erasure burst until the i-th packet is fully recovered.*

Hence the recovery delay differs from the standard delay of [1] (see sub-section II-A) in that it does not include the packets erased in the burst. An example illustrating this is given in Fig 2.

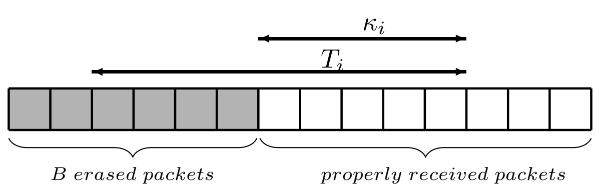

Figure 2: The recovery delay κ_i vs. the standard delay T_i.

It is a simple but helpful observation that the *average* recovery delay is equal to the average standard delay plus some constant depending only on B.

Proposition 1 *A code that can correct a burst erasure with length B and has an average delay $\bar{T}$ and an average recovery delay $\bar{\kappa}$ must satisfy*

$$\bar{T} = \bar{\kappa} + \frac{B-1}{2}. \tag{3}$$

Proof: According to the definition of κ_i we can write the following connection between κ_i and T_i

$$\kappa_i = T_i - (B - i) \tag{4}$$

and by calculating the average $\bar{\kappa}$ over all B erased packets we obtain the relation (3) ■

We will now see some simple examples for the profile of the recovery delay $\{\kappa_i\}_{i=1}^{B}$ in comparison to the profile of the standard delay $\{T_i\}_{i=1}^{B}$.

Example 1 *Constant delay: In this case the delay has a constant value, i.e., $\forall i \quad T_i = T$. Therefore the recovery delay of the oldest erased packet κ_1 is the shortest one, and according to (4) the recovery delay of the next erased packet κ_2 must be larger than κ_1 exactly by 1 in order to satisfy condition $T_1 = T_2 = T$. In general we can say that $\kappa_i = \kappa_{i-1} + 1$, so the packet with the longest recovery delay has $\kappa_B = T$. The resulting recovery delay profile $\{\kappa_i\}_{i=1}^{B}$ now follows, and illustrated pictorially in Figure 3.*

$$\begin{aligned}\{\kappa_i\}_{i=1}^{B} = \{\kappa_1 = T - (B-1)\,, \kappa_2 = T - (B-2), \ldots \\ , \kappa_{B-1} = T - 1, \kappa_B = T\}.\end{aligned} \tag{5}$$

The average recovery delay is clearly $\bar{\kappa} = T - \frac{B-1}{2}$.

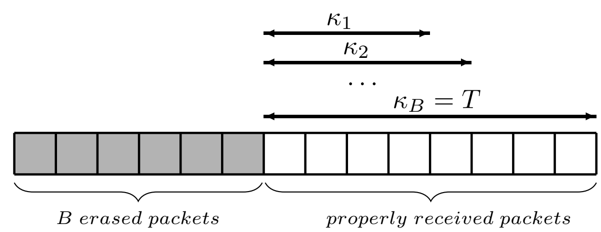

Figure 3: The profile of the recovery delay $\{\kappa_i\}_{i=1}^{B}$ for a constant delay.

Example 2 *MDS codes: When low-delay codes are constructed from $[n, k]$ MDS block codes as suggested in [1] (using techniques from [5]), packets are reconstructed after the arrival of k properly received packets[1] following the erasure burst of length $n - k$. Thus we have for this case that the recovery delay is constant $\kappa_i = \kappa = k \quad \forall i$. According to (4) we can find out that the profile of the standard delay is*

$$\begin{aligned}\{T_i\}_{i=1}^{B} = \{T_1 = k + (B-1)\,, T_2 = k + (B-2)\,, \ldots \\ , T_{B-1} = k + 1, T_B = k\},\end{aligned} \tag{6}$$

and the average standard delay is

$$\bar{T} = k + \frac{B-1}{2}. \tag{7}$$

III. Bounds on the Average Delay

The known delay lower bounds like the one in (1) only apply to constant delay, and it is not clear what limits exist for the average delay in the heterogeneous case. Hence in this section we seek such bounds on the average delay.

Theorem 2 *Let $\{T_i\}_{i=1}^{B}$ be the delay profile obtained for a given decoding instance following an erasure burst of length B. Then for a code with rate R the average delay $\bar{T}$ must satisfy*

$$\bar{T} \geq \max\left\{\frac{B+1}{2}, \frac{B}{2}\left(\frac{R}{1-R} + 1\right), \frac{R}{1-R} \cdot \frac{B+1}{2} + \frac{B-1}{2}\right\}. \tag{8}$$

[1]To simplify the discussion we ignore cases where a packet reconstruction is split between before and after the burst.

It can be seen in (8) that the lower bound is split to three rate intervals, applying to the rate intervals $R < \frac{1}{1+B}$, $\frac{1}{1+B} \leq R < \frac{1}{2}$ and $R \geq \frac{1}{2}$, respectively. Our proof will be divided into two parts: $R \geq \frac{1}{2}$ and $R < \frac{1}{2}$.
We first give the following useful lemma.

Lemma 3 *A rate R code carrying packets with S information symbols and P (causal[2]) parity symbols each has a recovery-delay profile $\{\kappa_i\}_{i=1}^{B}$ that satisfies*

$$\kappa_{\max} \geq \frac{BS}{P} = B \cdot \frac{R}{1-R}, \tag{9}$$

where $\kappa_{\max} \triangleq \max_{1 \leq i \leq B} \{\kappa_i\}$.

Proof: Because the encoder is causal, reconstructing all B erased packets requires at least BS/P packets following the burst. There is at least one packet i not yet fully decoded after $BS/P - 1$ packets following the burst, and this packet thus must have $\kappa_i \geq BS/P$. ∎

For convenience, Lemma 3 is stated for systematic codes, but a similar result follows for non-systematic codes as well. We are now ready to prove Theorem 2 for $R \geq \frac{1}{2}$.

Proof for $R \geq \frac{1}{2}$: In this proof we will use an inductive version of the proof of Lemma 3. From the lemma we already know that $\kappa_{\max}$ is at least $\left\lceil B \cdot \frac{S}{P} \right\rceil$, since every κ_i must be an integer. Now looking at the other $B-1$ packets, the maximal recovery delay among them is at least $\left\lceil (B-1) \frac{S}{P} \right\rceil$. Continuing this argument inductively we get the following lower bound for $\bar{\kappa}$

$$\bar{\kappa}_{R \geq \frac{1}{2}} \geq \frac{\left\lceil \frac{S}{P} \right\rceil + \cdots + \left\lceil (B-1) \cdot \frac{S}{P} \right\rceil + \left\lceil B \cdot \frac{S}{P} \right\rceil}{B} \geq$$

$$\frac{\frac{S}{P} + \cdots + (B-1) \cdot \frac{S}{P} + B \cdot \frac{S}{P}}{B} = \frac{R}{1-R} \cdot \frac{B+1}{2} \tag{10}$$

Note that we did not need the $R \geq \frac{1}{2}$ assumption to get the bound, but the assumption is necessary to make the second inequality tight by setting $\frac{S}{P} \in \mathbb{N}$. Now with the help of (3) we can write a lower bound for the average delay

$$\bar{T} = \bar{\kappa} + \frac{B-1}{2} \geq \frac{R}{1-R} \cdot \frac{B+1}{2} + \frac{B-1}{2}.$$

∎

Since the lower bound can be achieved only if $\frac{S}{P} \in \mathbb{N}$, we must choose the parameters S and P so they will satisfy $S = mP$, $m \in \mathbb{N}$, therefore in order to achieve the lower bound for $R \geq \frac{1}{2}$ the coding rate must be of the form $R = \frac{m}{m+1}$.

Proof for $R < \frac{1}{2}$: To make the proof simpler we will assume in the remainder of the proof that $\frac{P}{S} \in \mathbb{N}$ and $\frac{BS}{P} \in \mathbb{N}$. It can be shown that other cases give a strictly higher bound, but we omit this technical proof from this current presentation. For each received packet after the burst, we can reconstruct at most $\frac{P}{S}$ erased packets. So we can now replace the second inequality in (10) to obtain

$$\bar{\kappa}_{R < \frac{1}{2}} \geq \frac{\left\lceil \frac{S}{P} \right\rceil + \cdots + \left\lceil (B-1) \cdot \frac{S}{P} \right\rceil + \left\lceil B \cdot \frac{S}{P} \right\rceil}{B} \geq$$

[2]A causal parity only depends on information symbols from past and present packets.

$$\frac{\overbrace{1+1+\ldots+1}^{\frac{P}{S}\ times} + \overbrace{2+2+\ldots+2}^{\frac{P}{S}\ times} + \ldots + \overbrace{\frac{BS}{P} + \ldots + \frac{BS}{P}}^{\frac{P}{S}\ times}}{B}$$

$$= \frac{\frac{P}{S} \cdot \sum_{i=1}^{B \cdot \frac{S}{P}} i}{B} = \frac{P}{S} \cdot \frac{B \cdot \frac{S}{P} \left(B \cdot \frac{S}{P} + 1 \right)}{2B}$$

$$= \frac{B}{2} \cdot \frac{S}{P} + \frac{1}{2} = \frac{B}{2} \cdot \frac{R}{1-R} + \frac{1}{2}.$$

For the sub-interval $R < \frac{1}{1+B}$ the bound can be further tightened to $\bar{\kappa} \geq 1$, which follows trivially from the fact that each κ_i is at least 1. So altogether we obtain

$$\bar{\kappa}_{R < \frac{1}{2}} \geq \max \left\{ 1 \ , \ \frac{B}{2} \cdot \frac{R}{1-R} + \frac{1}{2} \right\}, \tag{11}$$

and with the help of (3) we get exactly the first two arguments of the lower bound (8). ∎

Figure 4 shows the lower bound for the average delay (solid) in comparison to the known lower bound for the constant delay (dashed). It can be seen that the two bounds coincide only at $R = \frac{1}{2}$.

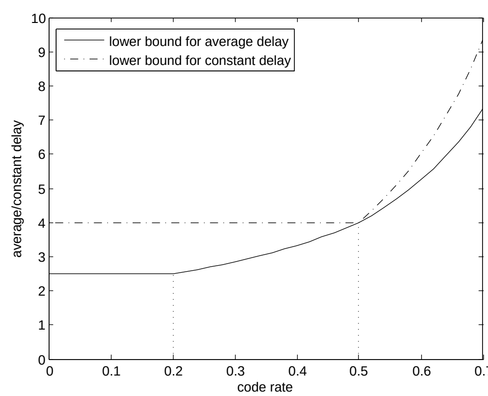

Figure 4: Lower bounds on decoding delay: average delay vs. constant delay, for $B = 4$.

IV. Constructions with Optimal Average Delay

After deriving lower bounds on the average delay, our objective is to find code constructions that attain these bounds. Similar to the bound derivation in Section III, the constructions of this section will split to codes with $R \geq \frac{1}{2}$ and codes with $R < \frac{1}{2}$.

A. Notations and definitions

A code construction is specified below through its encoder. The code is systematic, so each packet contains a systematic part with information symbols and a redundancy part with parity symbols. Each packet has an integer time index representing its order in the packet sequence. The encoder is causal, so parity symbols of a packet at time i are computed only from packets with time indices smaller than i. We now list some notations that will be used in the sequel.

- $s_j[i]$ – the j-th information symbol at time i, $s_j[i] \in F$ where F is a finite field.

- $s_j^l[i]$ – a vector of information symbols at time i containing $s_j[i], s_{j+1}[i], \ldots, s_l[i]$.
- $P\left\{s_{j_1}^{l_1}[i_1], s_{j_2}^{l_2}[i_2], \ldots, s_{j_k}^{l_k}[i_k]\right\}$ – is a parity-check function taking k symbol vectors of length $b \triangleq l_r - j_r + 1, \forall r$, and returning a symbol vector of the same size. P needs to satisfy the property that any of its arguments can be reconstructed from the output and the remaining $k-1$ arguments. One may think of P as the simple element-wise parity function over a finite field. We refer to P's k arguments and its output as a *parity group* (of size $k+1$).
- $\vec{x}[i]$ – the full packet at time i, including the information symbols and the parity symbols.

B. Construction for rates $R \geq \frac{1}{2}$

In this part we construct codes for $R \geq \frac{1}{2}$, which arguably is the more practically interesting case. Recall from Section III that achieving the lower bound (8) requires $R = \frac{m}{1+m}$ with $m \in \mathbb{N}$, so the construction will consider only rates of this form. For ease of presentation, we will restrict ourselves here to the special case $B = m+1$, but note that the construction can be easily adapted to general m, while keeping the same properties.

Construction 1 *For $B = m+1$ we define the following encoder specifying the packet output at time i*

$$\vec{x}[i] = \left(s_0^{mb-1}[i]\ ,\ \vec{x}^P[i]\right) \tag{12}$$

where

$$\vec{x}^P[i] = P\left\{\left\{s_{\langle i\rangle_m b}^{(\langle i\rangle_m+1)b-1}\left[\left\lfloor\frac{i}{m}\right\rfloor m - 1 - jB\right]\right\}_{j=0}^{m}\right\} \tag{13}$$

is the vector with b parity symbols. We use the notation $\langle i\rangle_m \triangleq i \bmod m$.

The encoder adds b parity symbols to every mb information symbols, yielding a code with rate $R = \frac{m}{1+m}$. It can be seen that the encoder of Construction 1 is systematic and causal. In Table I we can see an example of the encoding for $B = 3$ and $m = 2$ with $b = 3$.

The decoder *The packet $\vec{x}[i]$ with mb information symbols and b parity symbols is received by the decoder at time i. As long as no erasure occurred, the decoder uses only the information symbols. When the decoder identifies an erasure with burst-length B, it performs a reconstruction procedure during the next mB time units. Let us assume that the first packet received after the burst is at time index $\tilde{i}$, i.e., the erased packets are at time units $\tilde{i}-1, \tilde{i}-2, \ldots, \tilde{i}-B$. For each received packet at times $\tilde{i} \leq i \leq \tilde{i}+mB-1$ the decoder finds the missing argument in $\vec{x}^P[i]$ by finding j that satisfies the condition*

$$\left\lfloor\frac{i}{m}\right\rfloor m - 1 - jB \in \left\{\tilde{i}-1, \tilde{i}-2, \ldots, \tilde{i}-B\right\},\ j \in \{0, 1, \ldots, m\}. \tag{14}$$

Now the decoder uses $\vec{x}^P[i]$ and the known other m arguments of $\vec{x}^P[i]$ to reconstruct the following length b vector that was erased in the burst

$$s_{\langle i\rangle_m b}^{(\langle i\rangle_m+1)b-1}\left[\left\lfloor\frac{i}{m}\right\rfloor m - 1 - jB\right]. \tag{15}$$

Example 3 *For $B = 3$ and $m = 2$ the encoder is specified in Table I for $b = 3$. Suppose packets $\vec{x}[-3]$, $\vec{x}[-2]$ and $\vec{x}[-1]$ were erased, meaning the first received packet after the burst is $\vec{x}[0]$. Then, according to the specification of the decoder, the parity of $\vec{x}[0]$ will reconstruct $s_0^2[-1]$ from $s_0^2[-4]$ and $s_0^2[-7]$. Then the parity of $\vec{x}[1]$ will reconstruct $s_3^5[-1]$ from $s_3^5[-4]$ and $s_3^5[-7]$, and so forth until the parity of $\vec{x}[5]$ will reconstruct $s_3^5[-3]$ from $s_3^5[0]$ and $s_3^5[3]$. The resulting recovery delays of $\vec{x}[-3]$, $\vec{x}[-2]$ and $\vec{x}[-1]$ are 6, 4 and 2, respectively.*

$$\begin{aligned}
\vec{x}[-4] &= \left(s_0^5[-4]\ ,\ P\left\{s_0^2[-5]\ ,\ s_0^2[-8]\ ,\ s_0^2[-11]\right\}\right)\\
\vec{x}[-3] &= \left(s_0^5[-3]\ ,\ P\left\{s_3^5[-5]\ ,\ s_3^5[-8]\ ,\ s_3^5[-11]\right\}\right)\\
\vec{x}[-2] &= \left(s_0^5[-2]\ ,\ P\left\{s_0^2[-3]\ ,\ s_0^2[-6]\ ,\ s_0^2[-9]\right\}\right)\\
\vec{x}[-1] &= \left(s_0^5[-1]\ ,\ P\left\{s_3^5[-3]\ ,\ s_3^5[-6]\ ,\ s_3^5[-9]\right\}\right)\\
\vec{x}[0] &= \left(s_0^5[0]\ ,\ P\left\{s_0^2[-1]\ ,\ s_0^2[-4]\ ,\ s_0^2[-7]\right\}\right)\\
\vec{x}[1] &= \left(s_0^5[1]\ ,\ P\left\{s_3^5[-1]\ ,\ s_3^5[-4]\ ,\ s_3^5[-7]\right\}\right)\\
\vec{x}[2] &= \left(s_0^5[2]\ ,\ P\left\{s_0^2[1]\ ,\ s_0^2[-2]\ ,\ s_0^2[-5]\right\}\right)\\
\vec{x}[3] &= \left(s_0^5[3]\ ,\ P\left\{s_3^5[1]\ ,\ s_3^5[-2]\ ,\ s_3^5[-5]\right\}\right)\\
\vec{x}[4] &= \left(s_0^5[4]\ ,\ P\left\{s_0^2[3]\ ,\ s_0^2[0]\ ,\ s_0^2[-3]\right\}\right)\\
\vec{x}[5] &= \left(s_0^5[4]\ ,\ P\left\{s_3^5[3]\ ,\ s_3^5[0]\ ,\ s_3^5[-3]\right\}\right)
\end{aligned}$$

TABLE I: Example for the encoder of Construction 1 when $B = 3$, $m = 2$ and $b = 2$.

One may observe that decoding in Example 3 yields average delay of 5 (average recovery delay of 4), which satisfies the bound (8) with equality. We now turn to show that this optimality applies in more generality. To prove the precise optimality statement we first give the following definition.

Definition 3 *The burst's phase shift ϕ is defined as $\langle\tilde{i}\rangle_m$, where $\tilde{i}$ is the time unit of the first received packet after the length B burst.*

Theorem 4 *For a code specified by Construction 1, any burst with $\phi = 0$ has an average delay of*

$$\bar{T} = \frac{mB + m + B - 1}{2}. \tag{16}$$

Proof: Before arguing about delays, we prove the correctness of the decoder, i.e., that any length B burst with any phase shift can be successfully reconstructed. Examining (13), we see that all the time arguments are distant at least B time units from one another, so there can be up to one missing argument in every parity function. Moreover, we can see that only when $j = 0$ an argument of P can be erased in the same burst with the parity symbol of its group[3]. But according to (13), the same missing argument will appear again in another parity group (this time as $j = m$), which is distant mB time units from the erased one

$$\left\lfloor\frac{i+mB}{m}\right\rfloor m - 1 - mB \rightarrow \left\lfloor\frac{i}{m}\right\rfloor m - 1.$$

Therefore, all B erased packets will be successfully reconstructed. Now the calculation of the average delay will be based on finding which b sized vector is reconstructed at each time index. For the case $\phi = 0$, $\lfloor i/m\rfloor m - 1$ starts as $\tilde{i} - 1$

[3] When $j = 0$ the argument of P at time i points back $i - (\lfloor i/m\rfloor - 1) < B$ time units.

in $i = \tilde{i}$, and jumps by m every m received packets. Each m received packets we complete the reconstruction of a packet (whose index is defined by j). Therefore, the recovery delay profile for $\phi = 0$ equals

$$\{\kappa_i\}_{i=1}^{B} = \{m,\, 2m,\, 3m,\, \ldots\, ,\, mB\}\,, \tag{17}$$

and the average recovery delay and standard delay are

$$\bar{\kappa} = \frac{mB+m}{2} \iff \bar{T} = \frac{mB+m+B-1}{2}.$$

■

We reiterate that the average-delay lower bound of Theorem 2 applies to *every decoding instance*, and thus Theorem 4 implies the following optimality statement.

Corollary 5 *Construction 1 achieves the optimal average delay for an infinite number of decoding instances.*

Where infinite decoding instances refers to all instances with $\phi = 0$.

In the following we give the average delay of Construction 1 for a general phase shift (proof omitted).

Theorem 6 *For a given burst with a non-zero phase shift* $1 \leq \phi \leq m-1$, *a code specified by Construction 1 achieves the average delay*

$$\bar{T} = \frac{mB+m+B-1}{2} + \frac{(B-1)(m-\phi)}{B}. \tag{18}$$

To evaluate the delay performance of Construction 1, we plot in Figure 5 the min, max, and expected average delay, taken across the m possible phase shifts. We see that for every $B > 2$ Construction 1 gives superior delay compared to optimal constant-delay codes, including with respect to the maximum average delay among the phase shifts.

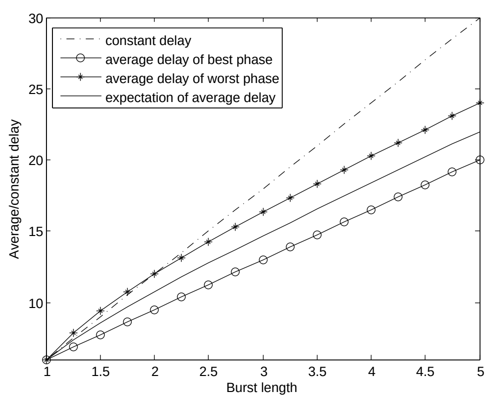

Figure 5: The average delay of Construction 1 compared to optimal constant delay. For Construction 1 three curves are plotted: the min, max, and expected among the m burst phase shifts.

C. Coding rate $R < \frac{1}{2}$

In this part we present codes with rate $\frac{1}{1+B} \leq R < \frac{1}{2}$ that achieve the optimal average delay. In this case, according to Section III the rate must be $R = \frac{1}{1+m}$, and also $\frac{BS}{P} \in \mathbb{N}$. In order to simplify the presentation, we show the construction for the special case $B = m\,(m+1)$, but it can be generalized to any B and m that satisfy $\frac{B}{m} \in \mathbb{N}$.

Construction 2 *for* $B = m\,(m+1)$, *we define the following encoder specifying the packet output at time* i

$$\begin{aligned}\vec{x}\,[i] = \Big(s_0^{mb-1}\,[i]\,, P\Big\{\big\{s_0^{mb-1}\,[i-1+j\,(m+1)]\big\}_{j\,\in\,J}\Big\}\,,\\ P\Big\{\big\{s_0^{mb-1}\,[i-2+j\,(m+1)]\big\}_{j\,\in\,J}\Big\}\,,\,\ldots\,,\\ P\Big\{\big\{s_0^{mb-1}\,[i-m+j\,(m+1)]\big\}_{j\,\in\,J}\Big\}\Big)\,,\end{aligned} \tag{19}$$

where

$$J = \begin{cases} \{0\,,\,-m\} & \text{if} \quad \langle i \rangle_m = 0 \\ -\,\langle i \rangle_m & \text{if} \quad \langle i \rangle_m \neq 0 \end{cases} \tag{20}$$

The encoder adds $m^2 b$ *parity symbols to every* mb *information symbols, so the code rate is* $R = \frac{1}{1+m}$.

Similarly to the case $R \geq \frac{1}{2}$, it can be seen that the code is systematic and causal. The decoding procedure is very similar to Construction 1, but now there are m parity functions in every packet, and the number of arguments in each parity function is not uniform.

Theorem 7 *For a code specified by Construction 2, any burst with any phase shift has an average delay*

$$\bar{T} = \frac{B}{2}\left(\frac{1}{m}+1\right). \tag{21}$$

The proof is omitted but it can be obtained by finding when each packet is being reconstructed. It can be seen that this is exactly the lower bound (8) for $R = \frac{1}{1+m}$.

Corollary 8 *Construction 2 has optimal average delay for* $\frac{1}{1+B} \leq R < \frac{1}{2}$ *for all decoding instances (independent on the phase shift).*

V. Conclusion

The most interesting open problem from this work is whether optimal average delay can be achieved for $R \geq 0.5$ in all erasure phases, and if not, what limits exist for the expected average delay across phases.

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