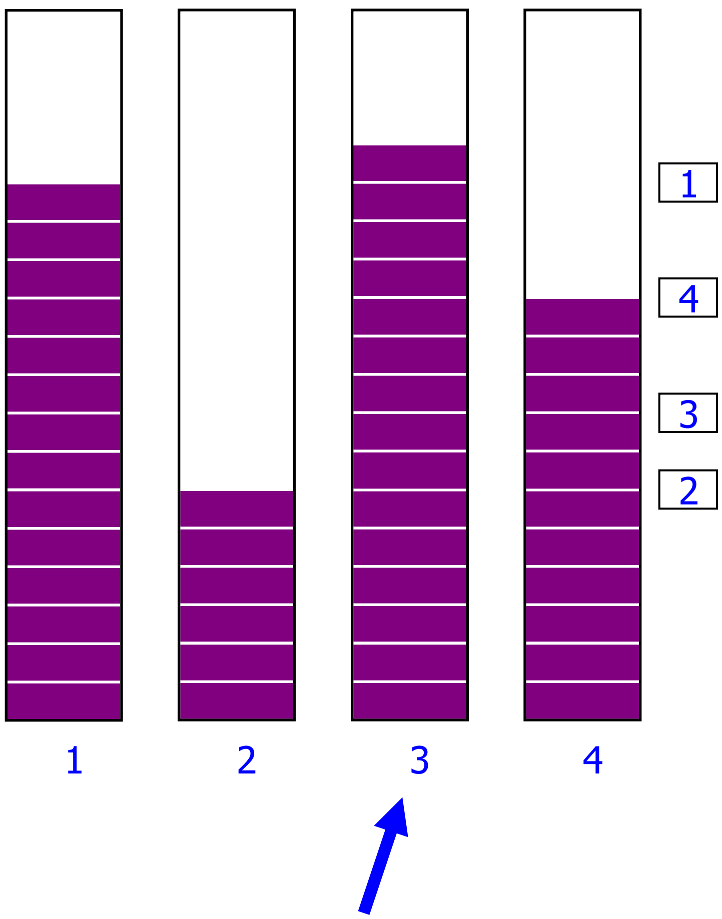
048704/236803
Seminar on Coding for Non-Volatile Memories

Rank Modulation



Ordered set of **n** cells

Assume discrete levels

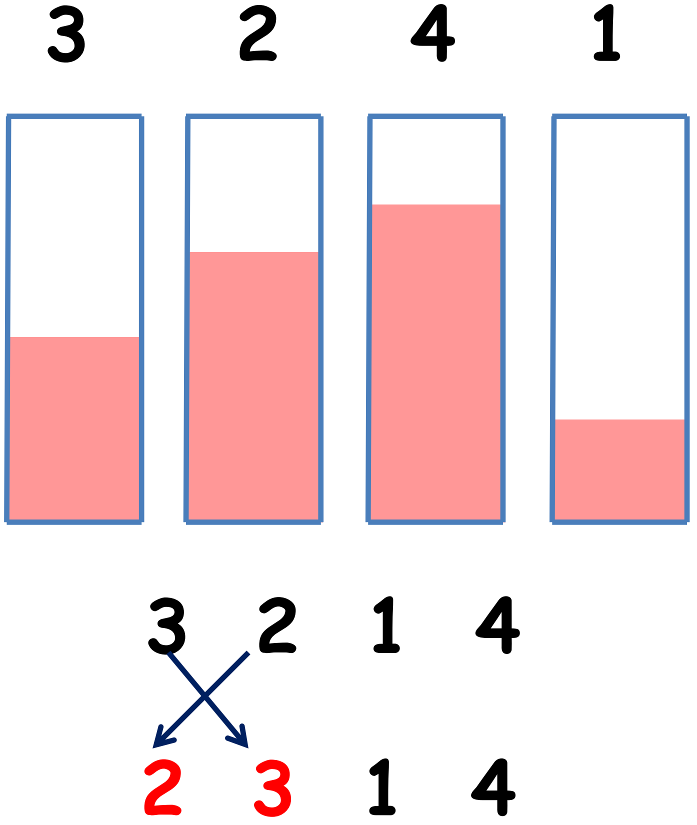
Relative levels define a **permutation**

Basic operation: **push-to-the-top**

Overshoot is not a concern

Writing is much faster

Increased reliability (data retention)


3 2 4 1
3 2 1 4
2 3 1 4

Kendall's Tau Distance

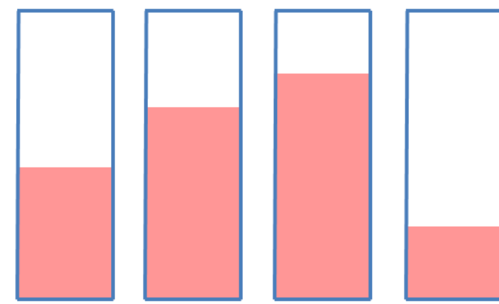
- For a permutation σ an **adjacent transposition** is the local exchange of two adjacent elements
- For $\sigma, \pi \in S_m$, $d_\tau(\sigma, \pi)$ is the **Kendall's tau** distance between σ and π

  = Number of adjacent transpositions to change σ to be π

  σ=2413 and π=2314

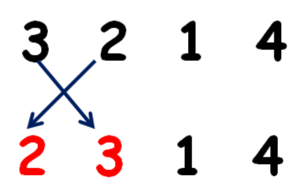
  2413 → 2143 → 2134 → 2314

  $d_\tau(\sigma,\pi) = 3$



It is called also the **bubble-sort** distance

The Kendall's tau distance is the number of pairs that do not agree in their order



Kendall's Tau Distance

- **Lemma:** Kendall's tau distance induces a metric on S_n
- The Kendall's tau distance is the number of pairs that do not agree in their order
- For a permutation σ, $W_T(\sigma) = \{(i,j) \mid i<j, \sigma^{-1}(i) > \sigma^{-1}(i) \}$
- **Lemma**: $d_T(\sigma,\pi) = |W_T(\sigma) \setminus W_T(\pi)| + |W_T(\pi) \setminus W_T(\sigma)|$
- $d_T(\sigma, id) = |W_T(\sigma)|$
- The maximum Kendall's tau distance is $n(n-1)/2$

ECCs for Kendall's Tau Distance

- **Goal**: Construct codes correcting a single error
- Assume k or k+1 is prime
- Encode a permutation in S_k to a permutation in S_{k+2}
- A code over S_{k+2} with k! codewords
  - **s**=$(s_1,\ldots s_k) \in S_k$ is the information permutation
  - set the locations of $k+1 \in Z_{k+1}$ and $k+2 \in Z_{k+2}$ to be
    $loc(k+1) = \Sigma_1^k (2i-1)s_i \pmod{m}$
    $loc(k+2) = \Sigma_1^k (2i-1)^2 s_i \pmod{m}$
    m=k if k is prime and m=k+1 is k+1 is prime
- **Ex**: k=7, **s**=(7613245)
  $loc(8) = 1\bullet7+3\bullet6+5\bullet1+7\bullet3+9\bullet2+11\bullet4+13\bullet5 = 3 \pmod{7}$
  $loc(9) = 1^2\bullet7+3^2\bullet6+5^2\bullet1+7^2\bullet3+9^2\bullet2+11^2\bullet4+13^2\bullet5 = 2 \pmod{7}$
  E(**s**) = (769183245)

ECCs for Kendall's Tau Distance

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    m=k if k is prime and m=k+1 is k+1 is prime
- **Ex**: k=3, m=3
  123 => 15423 ; loc(4)=1•1+3•2+5•3=1(mod3), loc(5)=1•1+9•2+25•3=1(mod3)
  132 => 13542 ; loc(4)=1•1+3•3+5•2=2(mod3), loc(5)=1•1+9•3+25•2=2(mod3)
  213 => 21543 ; loc(4)=1•2+3•1+5•3=2(mod3), loc(5)=1•2+9•1+25•3=2(mod3)
  231 => 52431 ; loc(4)=1•2+3•3+5•1=1(mod3), loc(5)=1•2+9•3+25•1=0(mod3)
  312 => 34512 ; loc(4)=1•3+3•1+5•2=1(mod3), loc(5)=1•3+9•1+25•2=2(mod3)
  321 => 35241 ; loc(4)=1•3+3•2+5•1=2(mod3), loc(5)=1•3+9•2+25•1=1(mod3)

ECCs for Kendall's Tau Distance

- A code over S_{k+2} with k! codewords
  - $\mathbf{s}=(s_1,\ldots s_k) \in S_k$ is the information permutation
  - set the locations of $k+1 \in Z_{k+1}$ and $k+2 \in Z_{k+2}$ to be
    $loc(k+1) = \Sigma_1^k(2i-1)s_i \pmod m$; $loc(k+2) = \Sigma_1^k(2i-1)^2 s_i \pmod m$
- **Theorem**: This code can correct a single Kendall's tau error.
- **Proof**: Enough to show that the Kendall's tau distance between every two codewords is at least 3
  - $\mathbf{s}=(s_1,\ldots s_k) \in S_k$, $\mathbf{u}=E(\mathbf{s})$; $\mathbf{t}=(t_1,\ldots t_k) \in S_k$, $\mathbf{v}=E(\mathbf{t})$
  - If $d_T(\mathbf{s},\mathbf{t})\geq 3$ then $d_T(\mathbf{u},\mathbf{v})\geq 3$ ✔
  - If $d_T(\mathbf{s},\mathbf{t})=1$, write $\mathbf{t}=(s_1,\ldots s_{i+1},s_i\ldots s_k)$, let $\delta = s_{i+1}-s_i$

  $loc_s(k+1)-loc_t(k+1)=(2i-1)s_i+(2i+1)s_{i+1}-(2i-1)s_{i+1}-(2i+1)s_i=2s_{i+1}-2s_i = 2\delta \pmod k$

  thus, they are **not** positioned in the same location. ✔

  $loc_s(k+2)-loc_t(k+2)=(2i-1)^2 s_i+(2i+1)^2 s_{i+1}-(2i-1)^2 s_{i+1}-(2i+1)^2 s_i$
  $=8is_{i+1}-8is_i = 8i\delta \pmod k$

  thus, they are **not** positioned in the same location either ✔.

ECCs for Kendall's Tau Distance

Proof (cont):

- $\mathbf{s}=(s_1,\ldots s_k) \in S_k$, $\mathbf{u}=E(\mathbf{s})$; $\mathbf{t}=(t_1,\ldots t_k) \in S_k$, $\mathbf{v}=E(\mathbf{t})$
- If $d_T(\mathbf{s},\mathbf{t})=1$, write $\mathbf{t}=(s_1,\ldots s_{i+1},s_i\ldots s_k)$, let $\delta = s_{i+1}-s_i$

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thus, they are **not** positioned in the same location either ✔.

- If $d_T(\mathbf{s},\mathbf{t})=2$, write $\mathbf{t}=(s_1,\ldots,s_{i+1},s_i,\ldots,s_{j+1},s_j,\ldots,s_k)$, $\delta_1=s_{i+1}-s_i$, $\delta_2=s_{j+1}-s_j$

  $loc_s(k+1)-loc_t(k+1)=2(\delta_1+\delta_2)(\text{mod } k)$

  $loc_s(k+2)-loc_t(k+2)=8i\delta_1 +8j\delta_2 \ (\text{mod } k)$

If $\delta_1+\delta_2$ is NOT a multiple of k then k+1 appears in different locations ✔

Otherwise, $\delta_2=-\delta_1 \ (\text{mod } k)$ and $loc_s(k+2)-loc_t(k+2)=8(i-j)\delta_1 \ (\text{mod } k)$,
so k+2 appears in different locations ✔

- If $d_T(\mathbf{s},\mathbf{t})=2$, write $\mathbf{t}=(s_1,\ldots,s_{i+2},s_i,s_{i+1},\ldots,s_k)$

  $loc_s(k+1)-loc_t(k+1)=(2i-1)s_i+(2i+1)s_{i+1}+(2i+3)s_{i+2}-(2i-1)s_{i+2}-(2i+1)s_i-(2i+3)s_{i+1})$
  $=4s_{i+2} -2s_{i+1}-2s_i \ (\text{mod } k)$

  $loc_s(k+2)-loc_t(k+2)=8((2i+1)s_{i+2}-(i+1)s_{i+1}-is_i)(\text{mod } k)$

Perfect t-Error-Correcting Codes

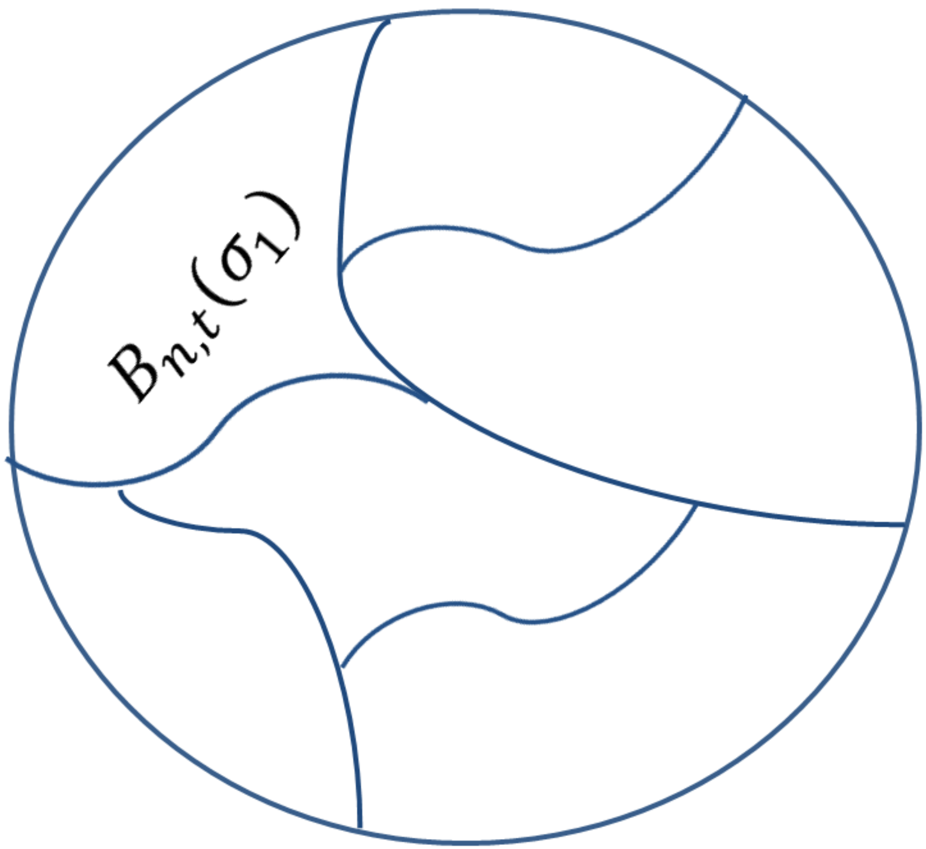
A perfect t-Error-Correcting Code:

A code $C \subset S_n$ s.t. for every $\pi \in S_n$ there exists **exactly** one codeword $\sigma \in C$ s.t. $d_K(\sigma, \pi) \leq t$.

The ball of radius t centered at $\sigma \in S_n$:

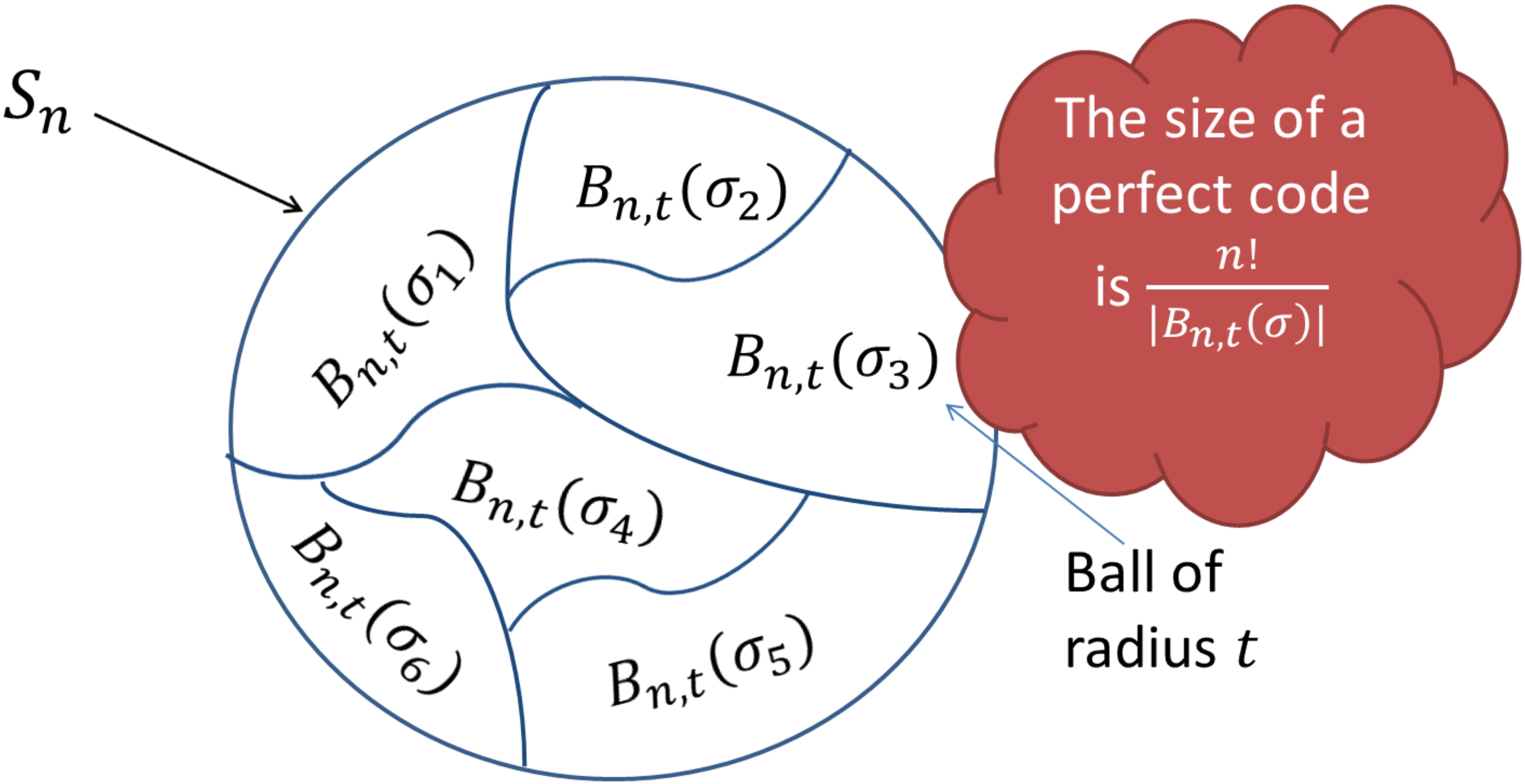
$$B_{n,t}(\sigma) = \{\pi \in S_n : d_K(\sigma, \pi) \leq t\}$$

Perfect t-Error-Correcting Codes



Codewords are the centers of the balls

Perfect t-Error-Correcting Codes



Codewords are the centers of the balls

Perfect t-Error-Correcting Codes

Examples:

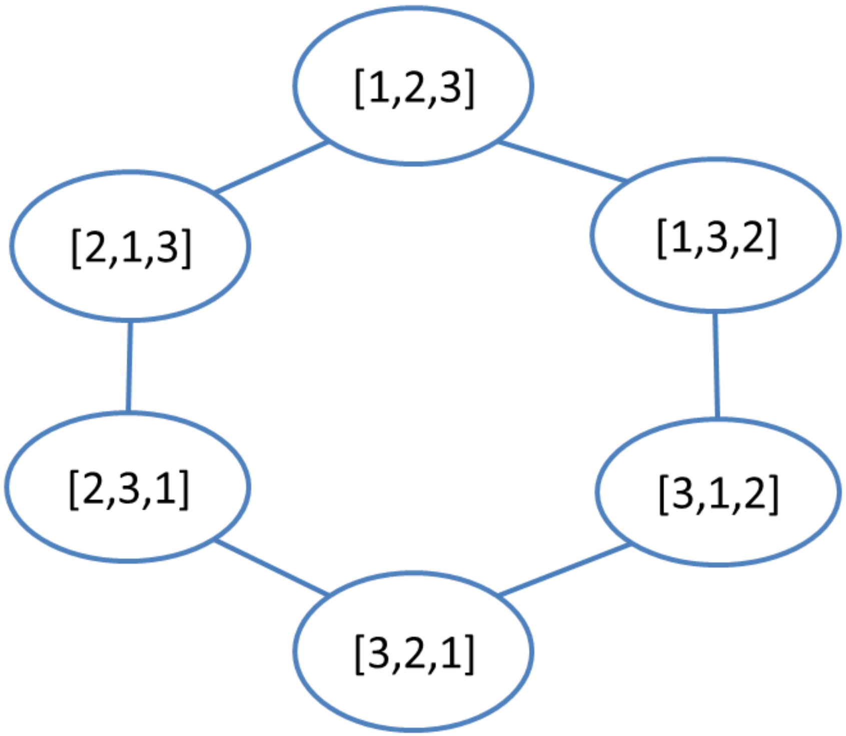
- $t = \binom{n}{2}$:

One codeword is a perfect $\binom{n}{2}$-error-correcting code in S_n.

Perfect t-Error-Correcting Codes

Examples:

- $n = 3, t = 1$:



Perfect t-Error-Correcting Codes

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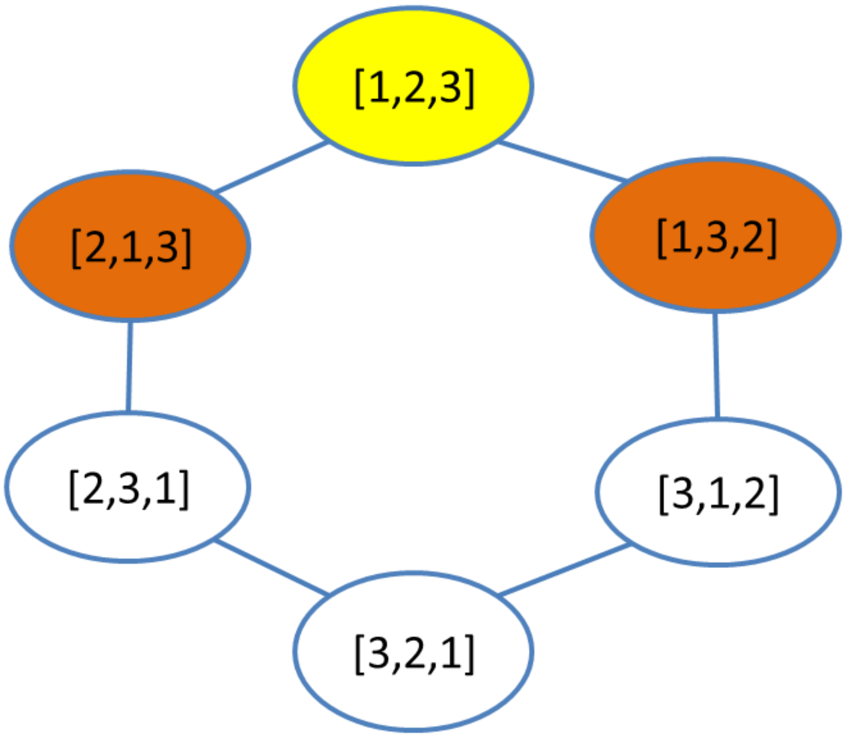

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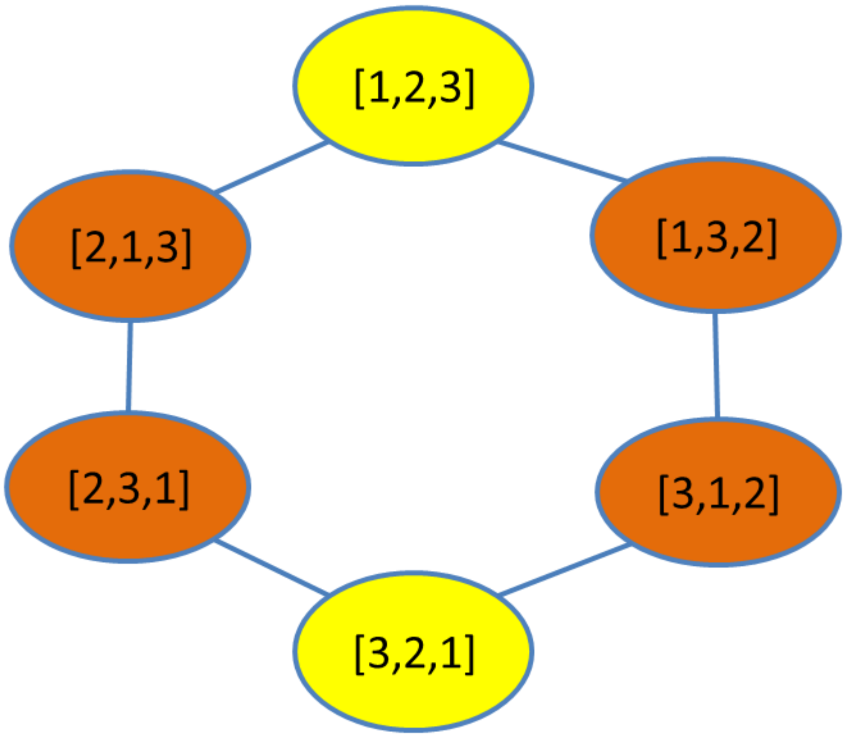
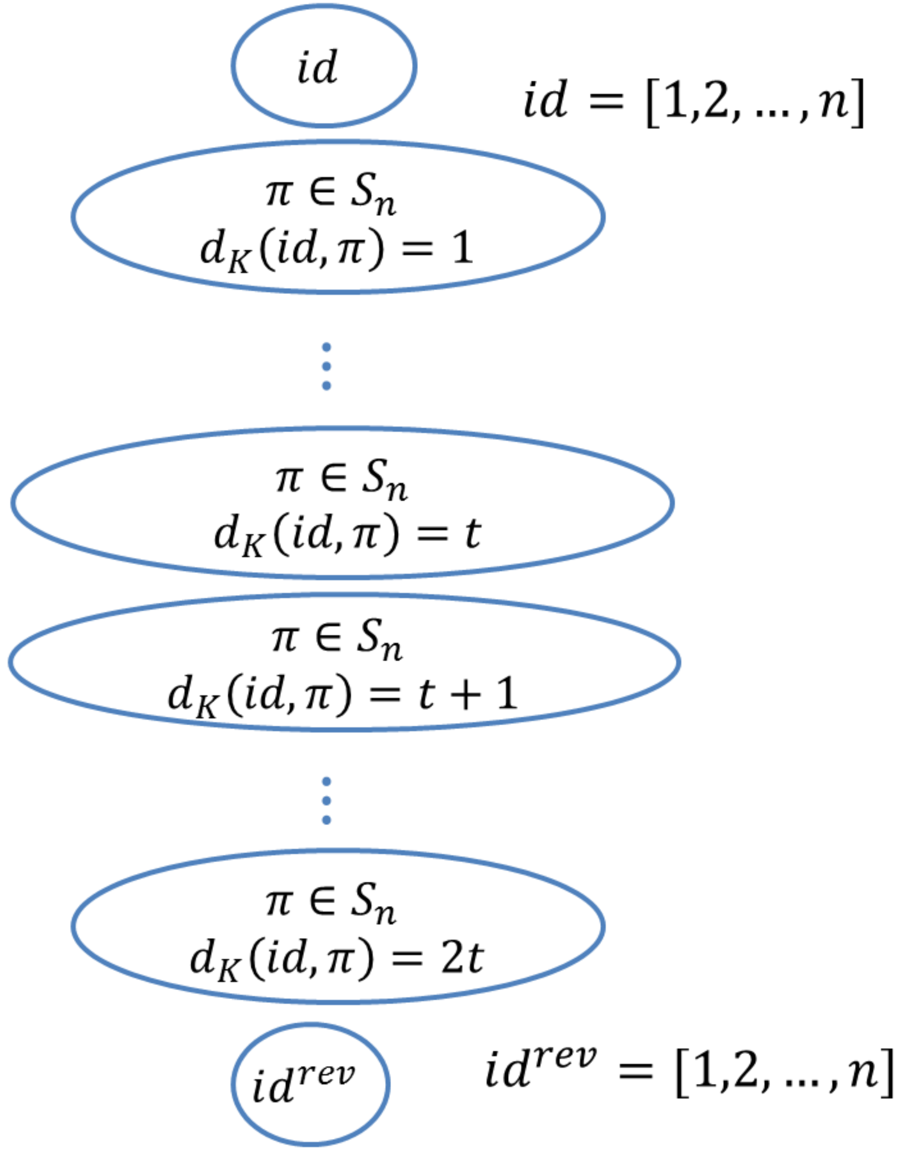


Perfect t-Error-Correcting Codes

Examples:

- $\binom{n}{2}$ is odd, $t = \frac{\binom{n}{2}-1}{2}$:



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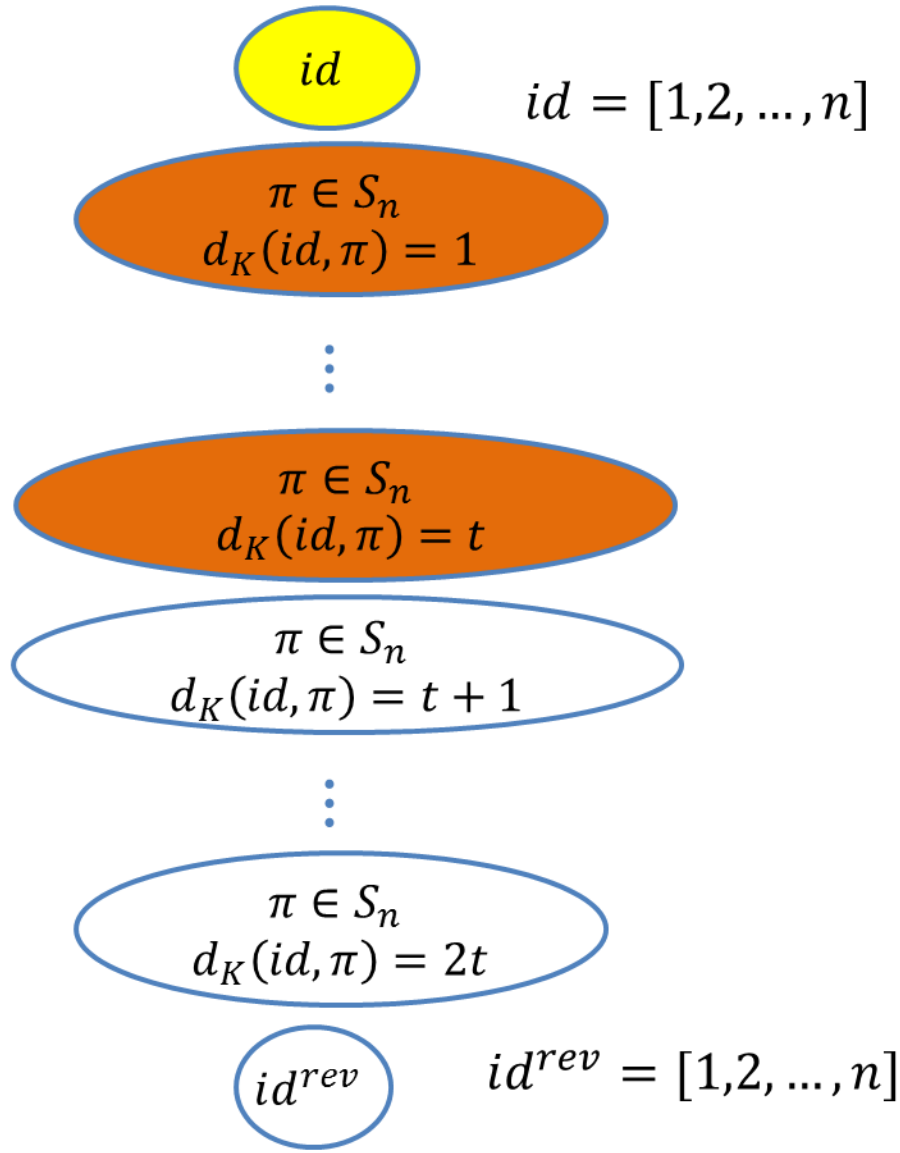

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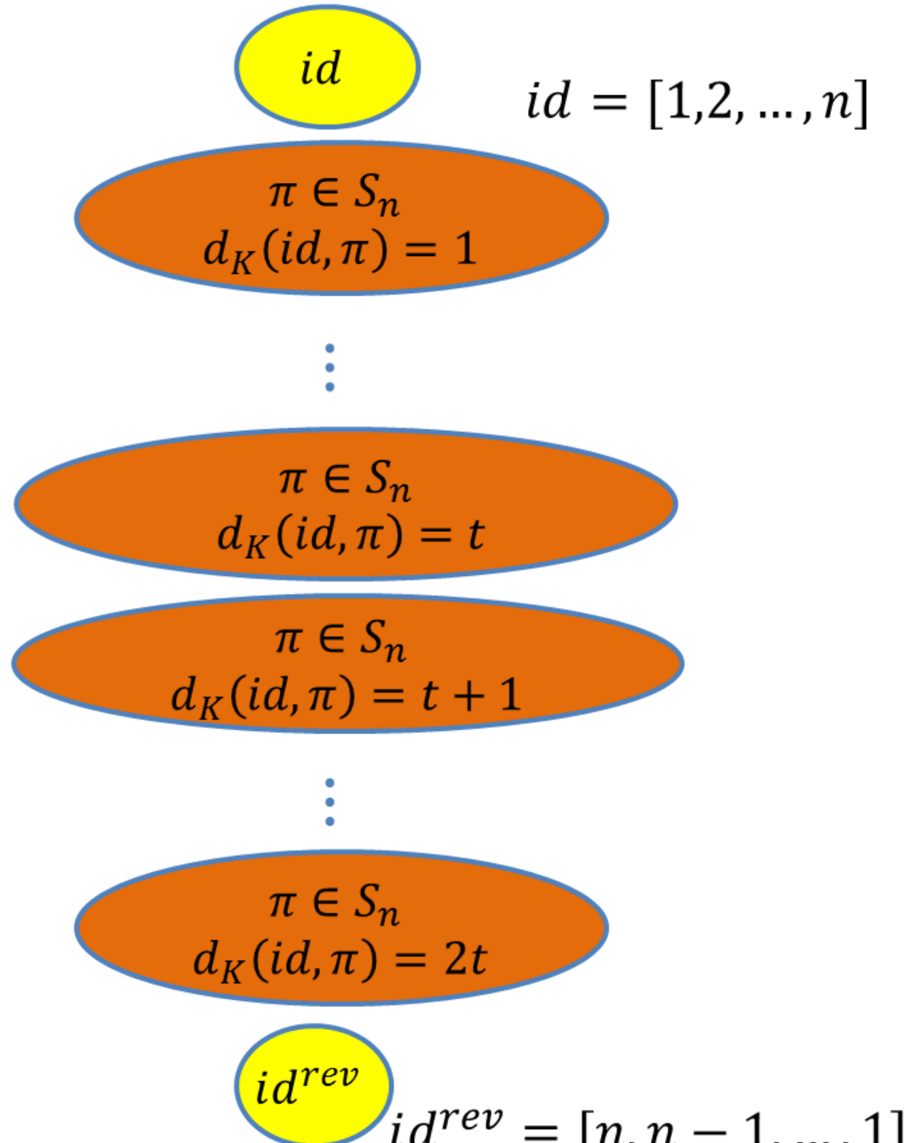

Perfect Single-Error-Correcting Code

Problem: Do perfect single-error-correcting codes exist?

- The size of such a code is $\frac{n!}{n} = (n-1)!$

Perfect Single-Error-Correcting Code

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- The size of such a code is $\frac{n!}{n} = (n-1)!$

Theorem: There is no perfect single-error-correcting code in S_n, $n > 3$ is a prime.

Proof

- Let $C \subset S_n$ be a perfect single-error-correcting code.

For all $1 \leq i \leq n$:

  - $S_i = \{\sigma \in S_n : \sigma(i) = 1\}$.
  - $C_i = C \cap S_i$.
  - $x_i = |C_i|$.

Proof

Counting elements of S_1:

$$(n-1)x_1 + x_2 = |S_1| = (n-1)!$$

Proof

Counting elements of S_1:

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Similarly,

$$x_{n-1} + (n-1)x_n = |S_n| = (n-1)!$$

and for all $2 \le i \le n-1$

$$x_{i-1} + (n-2)x_i + x_{i+1} = |S_i| = (n-1)!$$

Proof

$$\begin{pmatrix} n-1 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & n-2 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & n-2 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & n-2 & 1 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & n-2 & 1 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & n-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} (n-1)! \\ (n-1)! \\ \vdots \\ (n-1)! \end{pmatrix}$$

- The matrix is nonsingular for $n \geq 4 \Rightarrow$ unique solution

$x_i = \frac{(n-1)!}{n}$

Proof

$$\begin{pmatrix} n-1 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & n-2 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & n-2 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & n-2 & 1 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & n-2 & 1 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 & n-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} (n-1)! \\ (n-1)! \\ \vdots \\ (n-1)! \end{pmatrix}$$

- The matrix is nonsingular for $n \geq 4 \Rightarrow$ unique solution
  $x_i = \frac{(n-1)!}{n}$
- x_i is not an integer if $n > 3$ is a prime or if $n = 4$.
  No perfect single -error-correcting codes!!!