Seminar on Coding for Non-Volatile Memories

236803/048704 – CS/EE Departments, Technion

ASYMMETRIC, LIMITED-MAGNITUDE ERRORS

Coding: The General Problem

Some physical issue

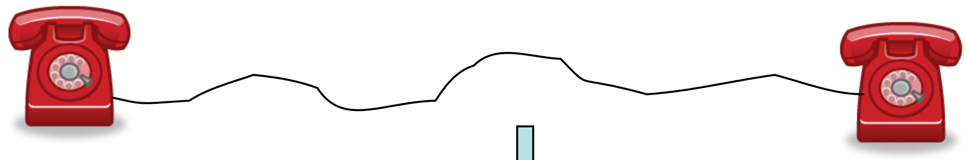
Data bits U

→ **Encoder** → **Decoder** →

Data bits Û

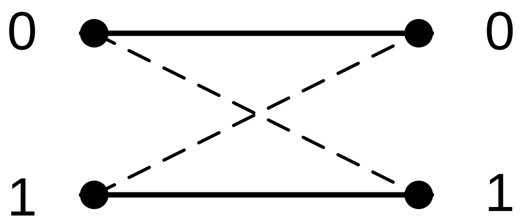
Find good codes that overcome physical issue and give U = Û

Classical Coding Theory

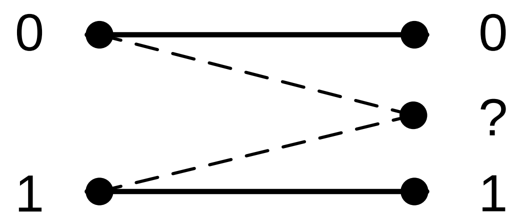


Shannon

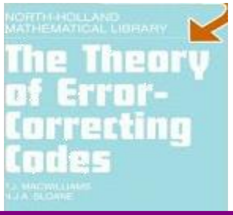
BSC



BEC

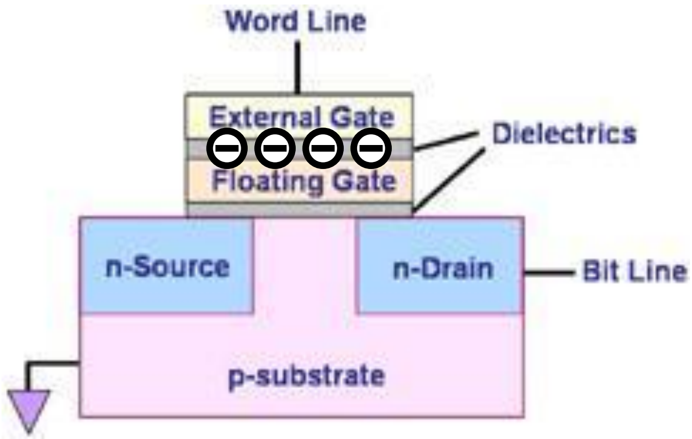


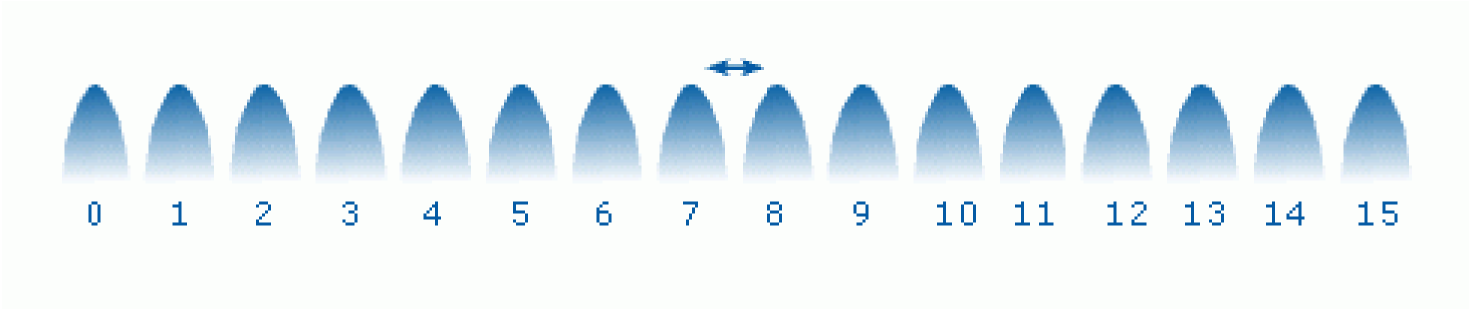
Hamming (distance)



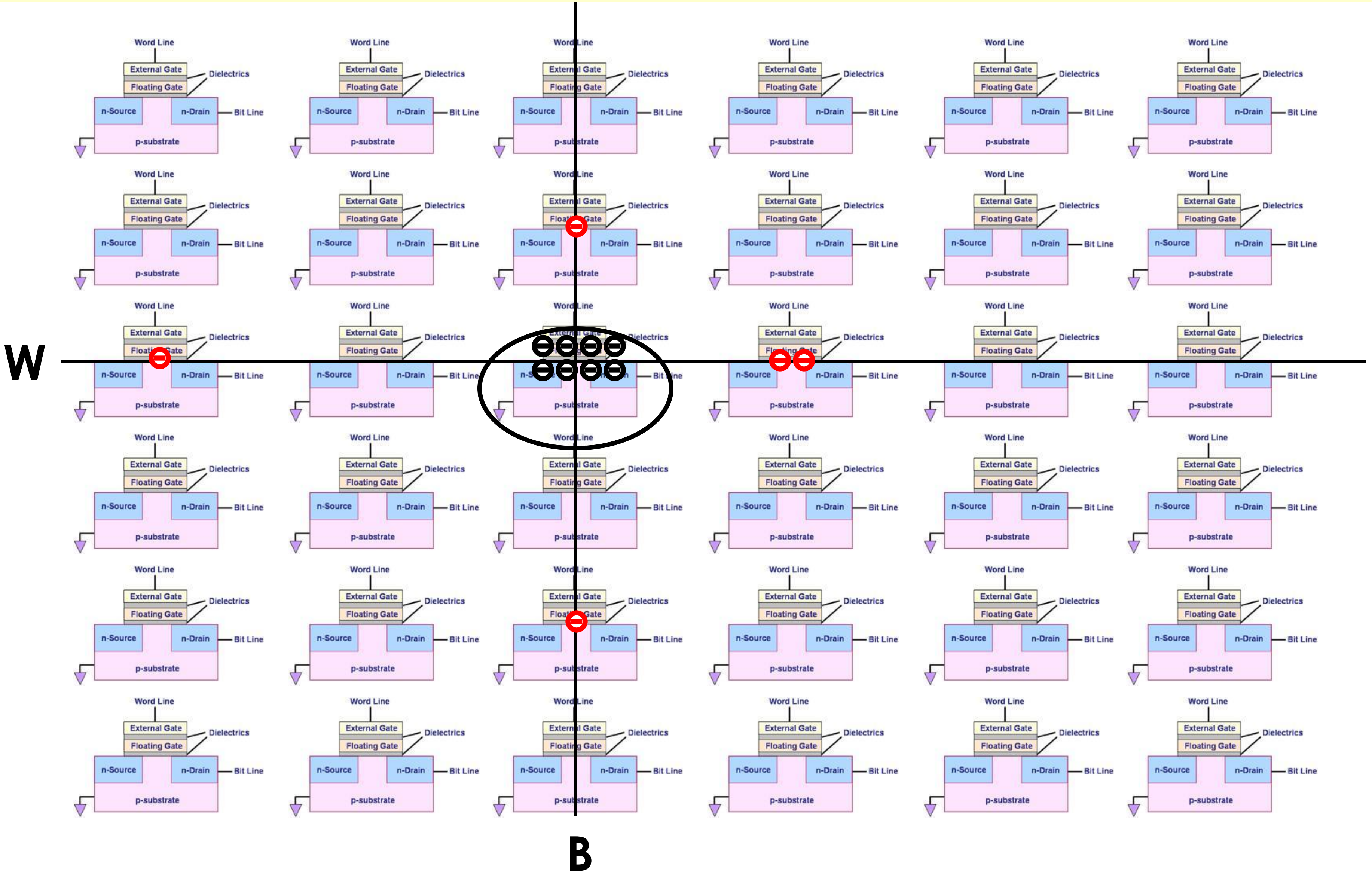
Codes for **Symmetric** **Symbol** Errors

Multi-Level Flash Storage

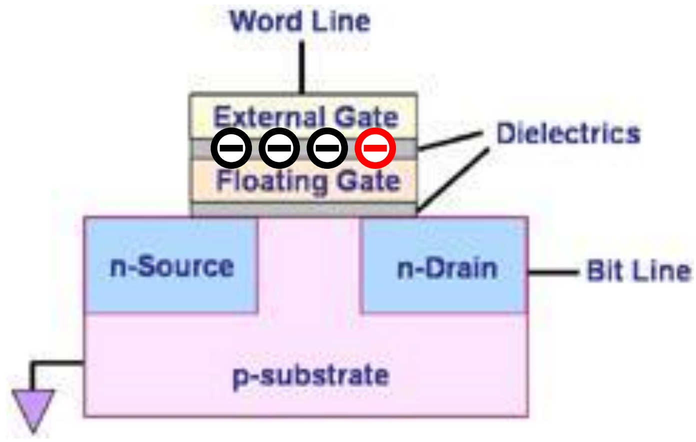


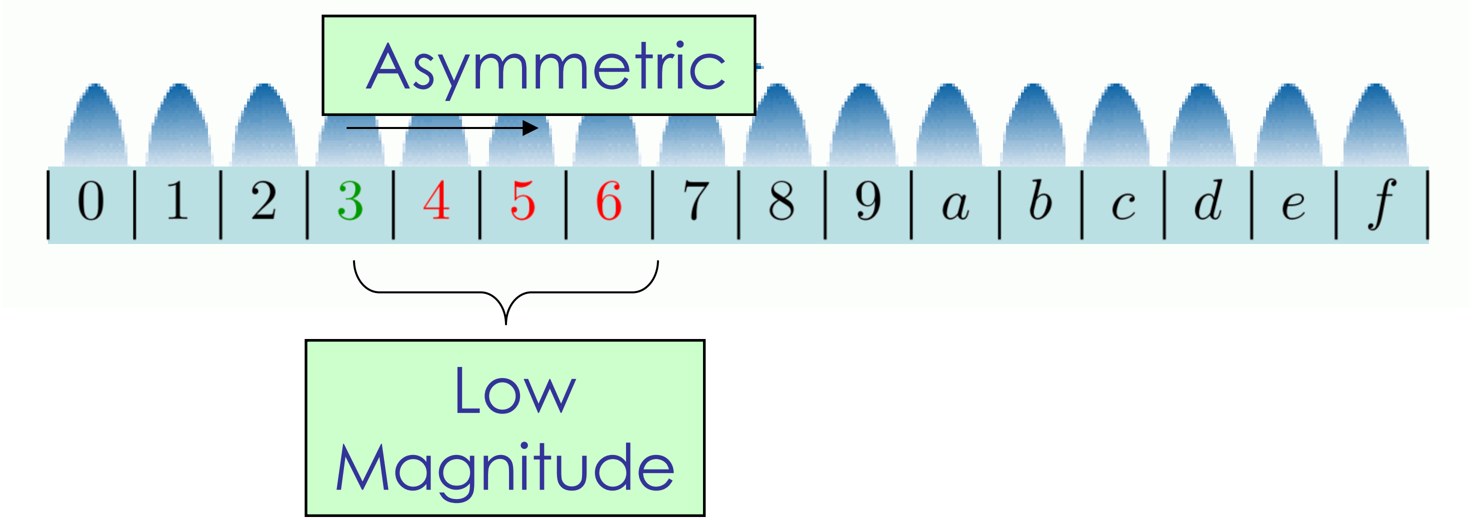


Dominant Flash Errors



Multi-Level Flash Errors





Asymmetric Limited-Magnitude Errors

Stored: $\vec{c} = [c_1, c_2, \ldots, c_n] \in Q^n$

Corrupted: $\vec{r} = [r_1, r_2, \ldots, r_n] \in Q^n$

Asymmetric **ℓ**imited-**M**agnitude Errors

$$0 \leq r_i - c_i \leq \ell, \quad w_H(\vec{r} - \vec{c}) \leq t$$

<u>Research Problem:</u>

Code $C \subset Q^n$ that corrects t $A\ell M$ errors

For each (n, ℓ, t)

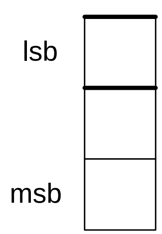
Example, $t=1, \ell=1$

18 information bits

$n = 7$ 8-ary cells (q=8)

0	1	1	0			
0	0	1	1	1	0	1
1	1	0	1	0	1	1

0	1	1	0	0	1	1	$\in \Sigma_H$, the Binary Hamming Code
0	0	1	1	1	0	1	
1	1	0	1	0	1	1	



$\rightarrow x_i \in Q = \{0, 1, \ldots, 7\}$

$\boldsymbol{x} =$

4	5	3	6	2	5	7	$\in C$

Decoding Example

4	5	3	6	2	5	7

A1M

4	5	4	6	2	5	7

0	1	0	0	0	1	1
0	0	0	1	1	0	1
1	1	1	1	0	1	1

*

Decode ∑

-1

+1

4	5	3	6	2	5	7

0	1	1	0	0	1	1
0	0	1	1	1	0	1
1	1	0	1	0	1	1

Systematic ALM Codes, $t = 1, \ell = 1$

$n = 15$ 4-ary cells (q=4)

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Encoding?

Sufficient and Necessary Condition

$n = 15$ 4-ary cells (q=4)

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Correcting 1 error? $\longleftrightarrow$ Unique syndromes to 1-error events

Algebraic Condition

n = *15* *4*-ary cells (q=4)

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Unique syndromes to 1-error events ⟷

NO! $(+1) \cdot s_1 = (+1) \cdot s_2$

NO! $(+1) \cdot s_1 = \mathbf{0}$

Systematic ALM Codes, $t = 1, \ell = 2$

$n = 12$ 5-ary cells (q=5)

$$H = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 \\ 1 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Encoding

Algebraic Condition, $\ell = \mathbf{2}$

n = *12* *5*-ary cells (q=5)

$$H = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 \\ 1 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

NO!

$$\alpha_1 \cdot \boxed{s_1} = \alpha_2 \cdot \boxed{s_2}$$

$$\alpha_1, \alpha_2 \in \{+1, +2\}$$

Decoding Example, $\ell = 2$

$n = 12$ 5-ary cells (q=5)

$$H = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 \\ 1 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Received syndrome:

$$s = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Was it a (+1) error? No.

It was a (+2) error. Where? $(+2)^{-1} \cdot 3 = 3 \cdot 3 = \boxed{4} \bmod 5$

Where exactly? $(+2)^{-1} \cdot 1 = 3 \cdot 1 = \boxed{3} \bmod 5$

Single Error Correction through $B_1[0,\ell]$ Sets

Definition – $B_1[0,\ell](q)$ set:
For a set B of size m, define

$$\Omega(B) = \{ib \bmod q \mid b \in B, i \in [0,\ell]\}.$$

Then B is a $B_1[0,\ell](q)$ set iff

$$|\Omega(B)| = \ell m + 1.$$

Examples:

$B_1[0,1](4) = \{1,2,3\}$

$\Omega(B) = \{0,1,2,3\}.$

$B_1[0,2](5) = \{1,4\}$ or $\{2,3\}$

$\Omega(B) = \{0,1,2,3,4\}.$

Construction from $B_1[0,\ell]$ Sets

Examples:

1) $B_1[0,1](4) = \{1,2,3\}$

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}$$

2) $B_1[0,2](5) = \{1,4\}$

$$H = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 \\ 1 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

2^{nd} Step Correctness

1) $B_1[0,1](4) = \{1,2,3\}$

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & 3 & \boxed{0 & 1 & 2 & 3} & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\alpha x_1 \neq \alpha x_2$$

2nd Step Correctness

2) $B_1\lceil 0,2 \rceil(5) = \{1,4\}$

$$H = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 \\ 1 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\alpha x_1 \neq \alpha x_2$$

$$\Downarrow$$

$$\alpha x \neq 0$$

$$\Downarrow$$

$$\gcd(\alpha, q) = 1$$

$$\Downarrow$$

$$\gcd(\ell!, q) = 1$$

Perfect $B_1[0,\ell]$ Sets

$$\Omega(B) = \{ib \bmod q \mid b \in B, i \in [0,\ell]\}.$$

Upper bound:

$$|\Omega(B)| = \ell m + 1 \leq q$$

perfect if equality

$B_1[0,1](4) = \{1,2,3\}$

$\Omega(B) = \{0,1,2,q-1\}.$

Perfect!

$B_1[0,2](5) = \{1,4\}$

$\Omega(B) = \{0,1,2,3,q-1\}.$

Perfect!

Extensions

- More than 1 error (t>1)
  - $B_t[0,\ell](q)$ sets, distinct sums of t products
- Symmetric limited magnitude errors
  - $B_1[-\ell,\ell](q)$ sets
- Unbalanced limited magnitude errors
  - $B_1[-\ell_1,\ell_2](q)$ sets